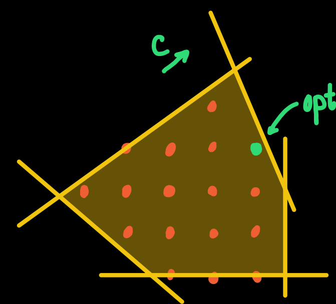
MA/AMA 514

Today: Integer programming (§8.1) & total unimodularity (8.2)

§8.1 Integer linear programming



An integer program is a problem of the form

$$\max\{c^T x : Ax \le b,\ x \in \mathbb{Z}^n\}$$

(maximize a linear function over the integer points $P \cap \mathbb{Z}^n$ of a polyhedron $P = \{x \in \mathbb{R}^n : Ax \le b\}$)

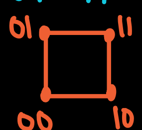
In general, these are NP-Hard to solve.

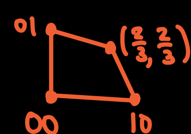
The associated LP gives an upper bound

$$\max\{c^T x : Ax \le b,\ x \in \mathbb{Z}^n\} \le \max\{c^T x : Ax \le b\}$$

Usually this bound is strict!

<u>Def</u> A polytope is <u>integral</u> (or "an integer polytope") if all of its vertices have integer coordinates.

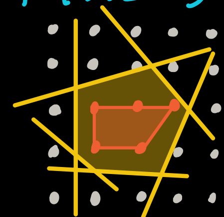
Ex's: 

$P_{matching}(G)$, $P_{perfect\,matching}(G)$ Non-ex: 

For integral polytopes P,

$$\max\{c^T x : x \in P\} = \max\{c^T x : x \in P \cap \mathbb{Z}^n\}$$

(Maximum is attained by some vertex of $P \Rightarrow$ in $P \cap \mathbb{Z}^n$)



Note: For any polytope P, $P \cap \mathbb{Z}^n$ is finite

$\Rightarrow P^I = \operatorname{conv}(P \cap \mathbb{Z}^n)$ also a polytope!

A polytope P is integral $\iff P^I = P$

<u>Thm 3.7</u> For any bipartite graph G,

$\{x \in \mathbb{R}^E : x_e \geq 0 \ \forall e \in E, \ \sum_{e \ni v} x_e = 1 \ \forall v \in V\}$

is integral (and so equals $P_{\text{perfect matching}}(G)$).

Proved by you in HW4!

Totally Unimodular Matrices

A matrix $A \in \mathbb{R}^{m \times n}$ is <u>totally unimodular</u> (TU) if every square submatrix has determinant $0, \pm 1$.

Ex: $A = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $\quad$ Non-ex: $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ Det(A) $= -2$!

<u>Thm 8.1</u> Let $A \in \mathbb{R}^{m \times n}$ be totally unimodular and $b \in \mathbb{Z}^m$. Every vertex of the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is an integer vector.

(Proof) Let z be a vertex of P. Recall that A_z is the submatrix of A given by the subset of rows a_i for which $a_i^T z = b_i$ and that $\text{rank}(A_z) = n$.

$\Rightarrow A_z$ has a nonsingular $n \times n$ submatrix A'.

(That is, there are n linearly indep. rows $a_{i_1}, \ldots, a_{i_n}$ of A s.t. $a_{i_k}^T z = b_{i_k}$ for all $k = 1, \ldots, n$. Take $A' = \begin{pmatrix} - a_{i_1} - \\ \vdots \\ - a_{i_n} - \end{pmatrix}$, $b' = \begin{pmatrix} b_{i_1} \\ \vdots \\ b_{i_n} \end{pmatrix}$)

Then $A'z = b' \in \mathbb{Z}^n$ and $z = (A')^{-1} b'$.

A totally unimodular $\Rightarrow \det(A') = \pm 1 \Rightarrow (A')^{-1} \in \mathbb{Z}^{n \times n}$

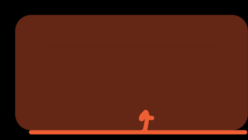
$\Rightarrow z = (A')^{-1} b' \in \mathbb{Z}^n$

<u>Lemma</u>: For $A \in \{0, \pm 1\}^{m \times n}$ the following operations do not change whether or not A is TU:

1) Multiplying a row/col by -1
2) Permuting rows/col
3) Adding a row or col with at most one ± 1 entry (and all other entries $= 0$)
4) Duplicating rows/cols
5) Transposing

You check!

Note: Some polyhedra don't have vertices

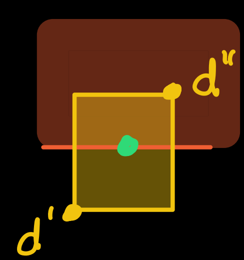


Call a polyhedron integer if $\max\{c^T x : x \in P\}$ is attained by an integer vector whenever it is finite

<u>Cor 8.1a</u> For totally unimodular $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is integer.

(Proof) Suppose $x^* \in P$ attains $\max\{c^T x : x \in P\}$.

Choose $d', d'' \in \mathbb{Z}^n$ s.t. $d' \leq x^* \leq d''$ (coord-wise)



Consider $Q = \{x \in \mathbb{R}^n : Ax \leq b,\ d' \leq x \leq d''\}$

$$= \left\{ x \in \mathbb{R}^n : \begin{bmatrix} A \\ -I \\ I \end{bmatrix} x \leq \begin{bmatrix} b \\ -d' \\ d'' \end{bmatrix} \right\}$$

The matrix $\begin{bmatrix} A \\ -I \\ I \end{bmatrix}$ must also be totally unimodular and $\begin{bmatrix} b \\ d' \\ d'' \end{bmatrix} \in \mathbb{Z}^{m+2n} \Rightarrow Q$ is an integer polytope

$\Rightarrow \max\{c^T x : x \in Q\}$ acheived by some integer vertex $\tilde{x}$.

Since $Q \subseteq P$, $c^T\tilde{x} = \max\{c^T x : x \in Q\} \leq \max\{c^T x : x \in P\} = c^T x^*$

But $x^* \in Q$, so $c^T\tilde{x} \geq c^T x^*$. Together this gives $c^T\tilde{x} = c^T x^*$.

That is, $\tilde{x} \in P \cap \mathbb{Z}^n$ acheives $\max\{c^T x : x \in P\}$.

<u>Cor</u>: If A is TU, $b \in \mathbb{Z}^m$ and $\ell, u \in \mathbb{Z}^n$ then each of the following polyhedra are integer:

1) $\{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$
2) $\{x \in \mathbb{R}^n : Ax \leq b,\ \ell \leq x \leq u\}$
3) $\{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$
4) $\{x \in \mathbb{R}^n : Ax = b,\ \ell \leq x \leq u\}$

Why?

1) $\underline{\begin{pmatrix} A \\ -I \end{pmatrix}} x \leq \begin{pmatrix} b \\ 0 \end{pmatrix}$

2) $\underline{\begin{pmatrix} A \\ -I \\ I \end{pmatrix}} x \leq \begin{pmatrix} b \\ -\ell \\ u \end{pmatrix}$

3) $\underline{\begin{pmatrix} A \\ -A \\ -I \end{pmatrix}} x \leq \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}$

4) $\underline{\begin{pmatrix} A \\ -A \\ -I \\ I \end{pmatrix}} x \leq \begin{pmatrix} b \\ -b \\ -\ell \\ u \end{pmatrix}$

If A is TU so are (1)-(4).

<u>Hoffman-Kruskal Thm</u> Let $A \in \mathbb{Z}^{m \times n}$.

A is totally unimodular

$\iff \forall b \in \mathbb{Z}^m$, $P = \{x \in \mathbb{R}^n_{\geq 0} : Ax \leq b\}$ is integer.

Talk about proof next time.

One important example of TU matrices:

<u>Prop</u>: Let $A \in \{0, \pm 1\}^{n \times m}$ be a matrix with at most one "1" and at most one "-1" in each column. Then A is TU.

(Proof) Let M be a $k \times k$ submatrix of A. We induct on k.

$(k=1)$ M = an entry of A $\Rightarrow \det(M) = M = 0, \pm 1$ ✓

$(k>1)$ <u>Case 1</u>: M has a column of 0's $\Rightarrow \det(M) = 0$

<u>Case 2</u>: M has a column w/ exactly one nonzero entry

$M = \begin{pmatrix} \pm 1 & v^T \\ 0 & \\ \vdots & \tilde{M} \\ 0 & \end{pmatrix} \Rightarrow \det(M) = \pm \det(\tilde{M})$ where $\tilde{M}$ is a sq. submatrix of size $k-1$ $\Rightarrow \det(M) = 0, \pm 1$

<u>Case 3</u>: Every column of M has one 1 and one -1

$\Rightarrow$ the vector $(1, .., 1)$ belongs to the left kernel of M

$\Rightarrow \det(M) = 0$