MA/AMA 514

Today: Application of Max Flow

Application: Elimination of sports teams

A sports team is eliminated if, regardless of the outcome of their remaining games, they can't finish with the most wins.

Ex:

Team	Wins	To Play
A	33	8
B	28	4

Even if B wins all remaining games it will have $32 < 33$ wins

$\rightarrow$ B eliminated

More complicated example

Team	Wins	To Play
A	33	8
B	29	4
C	28	7
D	27	5

Remaining to Play versus

A	B	C	D
*	1	6	1
1	*	0	3
6	0	*	1
1	3	1	*

It's possible for team B to tie team A if B wins all remaining games and A loses all remaining games. Then C must win its 6 games against A in which case C will have $\geq 28+6=34$ wins.

$\Longrightarrow$ B will have fewer wins than either A or C

$\Longrightarrow$ B eliminated

Q: Is there an easy way to tell when a team B is eliminated?

T = set of teams except B

w_i = current # wins for team i

r_{ij} = # games remaining between teams i and j.

$P = \{ \{i,j\} \subseteq T : i \neq j, \ r_{ij} > 0 \}$

M = # wins for B if they win all remaining games

Simple condition for elimination.

Take $\tilde{T} \subseteq T$.

Total # wins of teams in $\tilde{T}$ at the end of the season

$$\geq \sum_{i \in \tilde{T}} w_i + \underbrace{\sum_{\{i,j\} \subseteq \tilde{T}} r_{ij}}$$

every game between teams $i,j \in \tilde{T}$ is won by one of them

If this is strictly greater than $M \cdot |\tilde{T}|$ then

$$\text{avg \# wins for teams in } \tilde{T} \geq \frac{\sum_{i \in \tilde{T}} w_i + \sum_{\{i,j\} \subseteq T} r_{ij}}{|\tilde{T}|} > M$$

$\Longrightarrow$ Some team in $\tilde{T}$ ends with more than M wins

<u>Thm</u>: B is eliminated $\iff \exists \tilde{T} \subseteq T$ with

$$\sum_{i \in \tilde{T}} w_i + \sum_{\{i,j\} \subseteq \tilde{T}} r_{ij} > M \cdot |\tilde{T}|.$$

(Proof) ($\Leftarrow$) Above!

($\Rightarrow$) Idea: if B is not eliminated, then for some possible outcomes of remaining games, B finishes with the most wins

Take y_{ij} = # wins of i over j in remaining r_{ij} games.

B is not eliminated $\Leftrightarrow \exists\, y_{ij}$ with

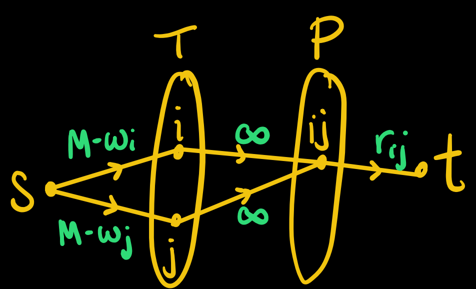
$$(*)\begin{cases} y_{ij} + y_{ji} = r_{ij} \quad \forall\, ij \in P & \leftarrow \text{each remaining game won by one of the teams} \\ w_i + \displaystyle\sum_{\{ij\} \in T \cap P} y_{ij} \le M \quad \forall\, i \in T & \leftarrow \text{team } i \text{ ends with } \le M \text{ wins} \\ y_{ij} \ge 0 \ \text{ and } \ y_{ij} \in \mathbb{Z} \quad \forall\, ij \in P & \leftarrow \#\text{games} \ge 0 \text{ and integer} \end{cases}$$

Create directed graph $D=(V,A)$ with

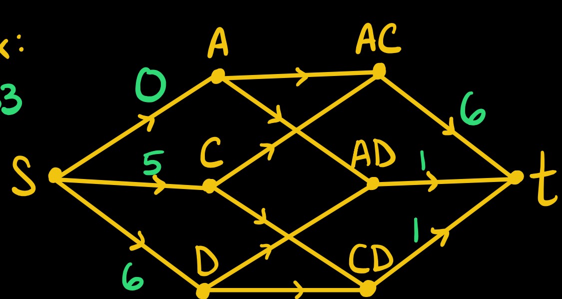
vertices $V = T \cup P \cup \{s,t\}$

arcs $A = \{s \to i : i \in T\} \cup \{i \to ij : {i \in T \atop ij \in P}\} \cup \{ij \to t : ij \in P\}$

capacities $\quad M - w_i \qquad\qquad \infty \qquad\qquad r_{ij}$



In Ex: $M = 33$



Claim: D has an s-t flow under c of value $\sum_{ij \in P} r_{ij}$

$\Leftrightarrow$ $(*)$ has a solution ($\Leftrightarrow$ B not eliminated!)

(Proof of Claim) ($\Rightarrow$) If D has such a flow, then it has an integer flow $f: A \to \mathbb{Z}_{\geq 0}$. Take y_{ij} = flow on arc $i \to ij$.

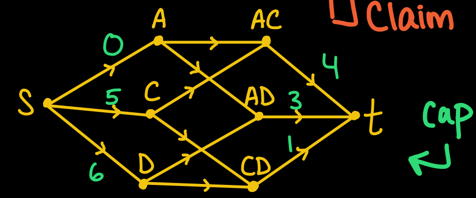
Check: This gives solution to $(*)$!

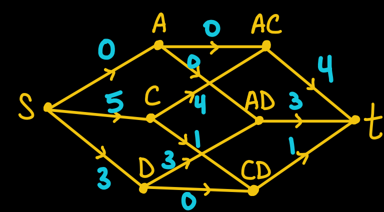
($\Leftarrow$) Given solution $(y_{ij})_{ij}$ to $(*)$ assign flow y_{ij} to $i \to ij$ and extend to other arcs by conservation!

$$f(s \to \omega_i) = \sum_{\{i,j\} \in T \cap P} y_{ij} \quad (\leq M - \omega_i) \quad \text{and} \quad f(ij \to t) = r_{ij} \quad (= y_{ij} + y_{ji})$$

□ Claim

Team	Wins	To Play	Remaining to Play versus A	B	C	D
A	33	8	*	1	4	3
B	29	4	1	*	2	1
C	28	7	4	2	*	1
D	27	5	3	1	1	*





B eliminated $\Rightarrow$ max s-t flow on D has value $< \sum_{ij \in P} r_{ij}$.

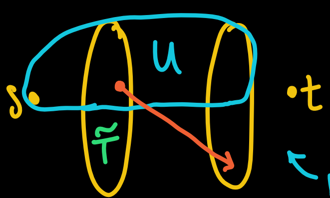
By Max-Flow Min-Cut Thm,

$\exists$ s-t cut $U \subseteq V$ with $c(\delta^{out}(U)) < \sum_{ij \in P} r_{ij}$

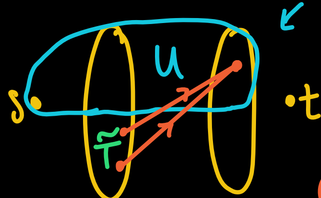
Take $\tilde{T} = T \setminus U$.

equiv: $i \in U$ and $j \in U$

Claim: $U = \{s\} \cup (T \setminus \tilde{T}) \cup \{ij \in P : i \notin \tilde{T} \text{ or } j \notin \tilde{T}\}$



neither happens



(Proof) If $i \notin \tilde{T}$ or $j \notin \hat{T}$ and $ij \notin U$ then $\exists$ edge of ∞-capacity in $\delta^{out}(U)$. $\divideontimes$

If $ij \in U$ and $i, j \in \tilde{T}$, then

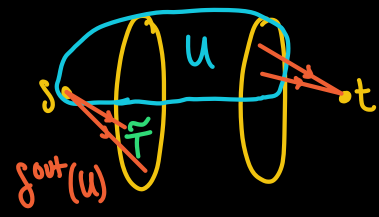
$$c(\delta^{out}(U \setminus ij)) = c(\delta^{out}(U)) - r_{ij} < c(\delta^{out}(U)) \quad \divideontimes$$

□ Claim

Capacity of $\delta^{out}(U)$

$$= \sum_{i \in \tilde{T}} (M - w_i) + \sum_{\substack{ij \in P \\ \{i,j\} \not\subseteq \tilde{T}}} r_{ij} < \sum_{ij \in P} r_{ij}$$

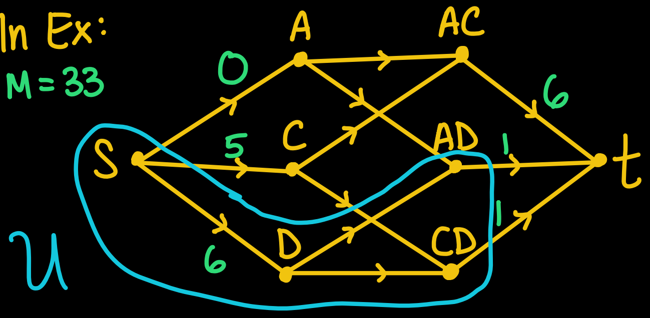
by assumption



$$\Rightarrow M \cdot |\tilde{T}| < \sum_{i \in \tilde{T}} w_i + \sum_{\substack{ij \in P \\ \{i,j\} \subseteq \tilde{T}}} r_{ij}$$

$\square$ Thm

In Ex:
$M = 33$



Max flow $= 7 < 8 = 6+1+1$

Min-cut : $U = \{s, D, AD, CD\}$

$\tilde{T} = \{A, C\}$

$\Rightarrow$ avg. # wins of A, C > max # wins by B