MA/AMA 514

Today: §4.5, §4.6

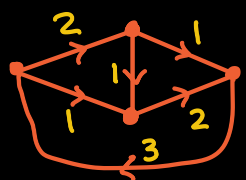
§4.5 Circulations

A function $f: A \to \mathbb{R}$ is a circulation on a directed graph $D=(V,E)$ if for every $v \in V$,

$$\sum_{a \in \delta^{in}(v)} f(a) = \sum_{a \in \delta^{out}(v)} f(a).$$

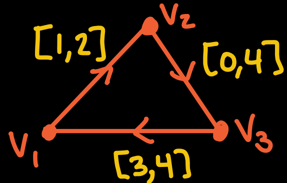
← flow conserved at every vertex!

Ex:



Question: Given $d: A \to \mathbb{R}$, $c: A \to \mathbb{R}$, does $\exists$ a circulation f with $d(a) \le f(a) \le c(a)$ $\forall a$?

Ex:



no circulation possible!

Need $f(v_1 \to v_2) = f(v_3 \to v_1)$ for net flow 0 at v_1

But $f(v_1 \to v_2) \le 2 < 3 \le f(v_3 \to v_1)$

Hoffman's Circulation Thm Let $D=(V,A)$ be a directed graph and $d, c: A \to \mathbb{R}$ with $d(a) \le c(a)$ $\forall a \in A$. Then $\exists$ a circulation f with $d(a) \le f(a) \le c(a)$ $\forall a$ $\iff$ for all $U \subseteq V$

$$\sum_{a \in \delta^{in}(U)} d(a) \le \sum_{a \in \delta^{out}(U)} c(a).$$

(Proof) ($\Rightarrow$) For any circulation f and subset $U \subseteq V$,

$$\sum_{a \in \delta^{in}(U)} d(a) \le \sum_{a \in \delta^{in}(U)} f(a) = \sum_{a \in \delta^{out}(U)} f(a) \le \sum_{a \in \delta^{out}(U)} c(a).$$

($\Leftarrow$) For a function $f: A \to \mathbb{R}$, define

$$\text{loss}_f(v) = f(\delta^{out}(v)) - f(\delta^{in}(v)) \quad \begin{pmatrix}\text{net loss of} \\ \text{flow at } v\end{pmatrix}$$

Suppose $f: A \to \mathbb{R}$ minimizes $\sum_{v \in V} |\text{loss}_f(v)|$ over all functions f with $d \le f \le c$. Let

$S = \{v \in V : \text{loss}_f(v) < 0\}$, $T = \{v \in V : \text{loss}_f(v) > 0\}$.

If $S = \emptyset$, then $T = \emptyset$ and f is a circulation.

Otherwise, define the directed graph $D_f = (V, A_f \cup B_f)$ where $A_f = \{a \in A : f(a) < c(a)\}$, $B_f = \{\overleftarrow{a} : a \in A, d(a) < f(a)\}$.

Claim: D_f does not have a path from S to T.

If $P: s \xrightarrow{a_1} v_1 \xrightarrow{a_2} \cdots \xrightarrow{a_k} t$ is a path in D_f from $s \in S$ to $t \in T$, define $f': A \to \mathbb{R}$ by $f'(a) = \begin{cases} f(a) + \alpha & \text{if } a \in A_f \cap P \\ f(a) - \alpha & \text{if } a \in B_f \cap P \\ f(a) & \text{otherwise.} \end{cases}$

where $\alpha > 0$ is sufficiently small.

Note: $\text{loss}_{f'}(v) = \text{loss}_f(v)$ for $v \ne s, t$, $\text{loss}_{f'}(s) = \text{loss}_f(s) + \alpha$, $\text{loss}_{f'}(t) = \text{loss}_f(t) - \alpha$

Then $d \le f' \le c$ and $\|\text{loss}(f')\|_1 < \|\text{loss}(f)\|_1$ $\;\Rightarrow\!\Leftarrow$ $\square$ Claim

Let U = set of vertices in D_f reachable from S.

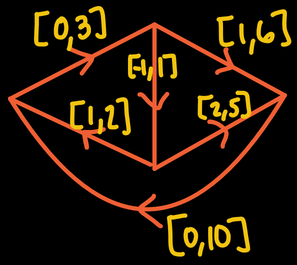
For $a \in \delta^{out}(U)$, $a \notin A_f \cup B_f \Rightarrow f(a) = c(a)$

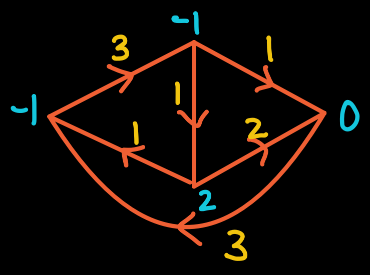
For $a \in \delta^{in}(U)$, $\overleftarrow{a} \notin A_f \cup B_f \Rightarrow f(a) = d(a)$.

$\Rightarrow c(\delta^{out}(U)) - d(\delta^{in}(U)) = f(\delta^{out}(U)) - f(\delta^{in}(U)) = \text{loss}_f(U) = \text{loss}_f(S) < 0$.

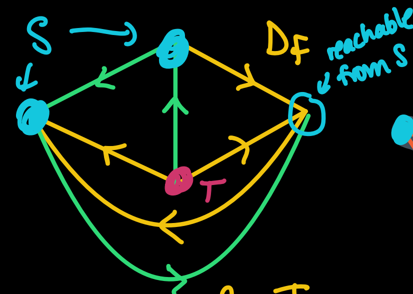
By Claim $U \cap T = \emptyset \Rightarrow \text{loss}_f(u) = 0$ for $u \in U \setminus S$.

Ex:

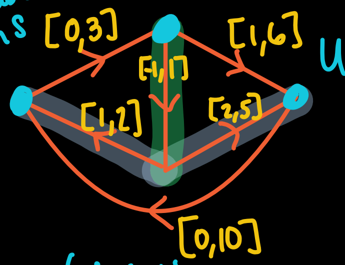




$\mathrm{loss}_f(v)$



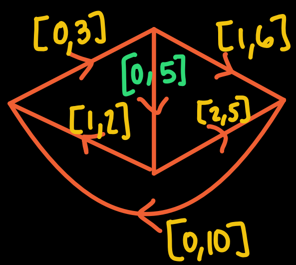
no path $S \to T$

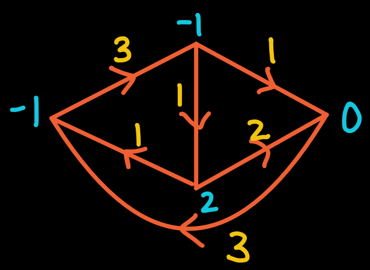


$d(\delta^{in}(u)) = 3$

$c(\delta^{out}(u)) = 1$

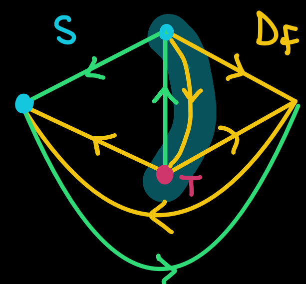
Ex: Augmenting path from claim?



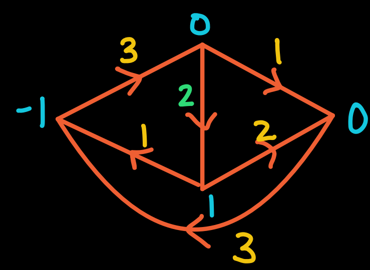


$\mathrm{loss}_f(v)$

$\|\mathrm{loss}(f)\|_1 = 4$



S-T path!



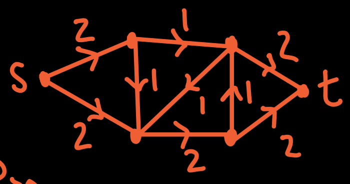
$\mathrm{loss}_f(v)$

$\|\mathrm{loss}(f')\|_1 = 2$

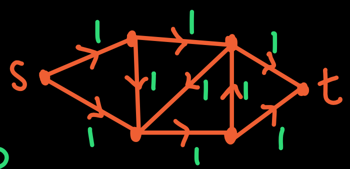
§4.6 Minimum cost flows

Given $D=(V,A)$, $s,t \in V$, capacities $c: A \to \mathbb{R}_{>0}$ and costs $k: A \to \mathbb{R}_{>0}$, find a flow $f: A \to \mathbb{R}$ of minimum cost among all flows under c of maximum value where

$$\mathrm{cost}(f) = \sum_{a \in A} k(a) f(a)$$

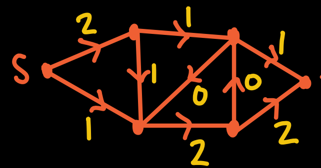
Ex: 

$c: A \to \mathbb{R}_{>0}$

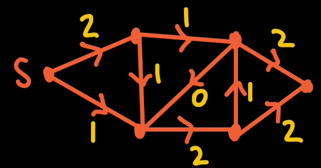


$K: A \to \mathbb{R}_{>0}$

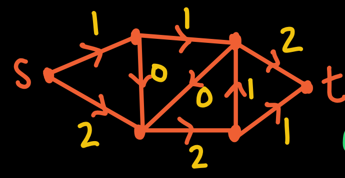
Max flows under c:



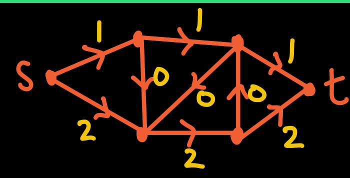
cost 10



cost 11



cost 10

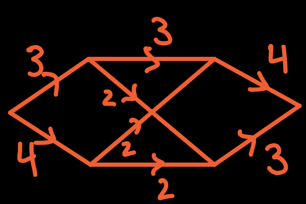


cost 9

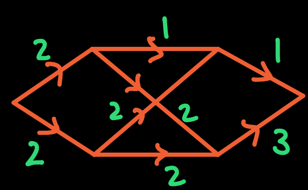
min cost max flow

For details on how to find combinatorially, see §4.6.

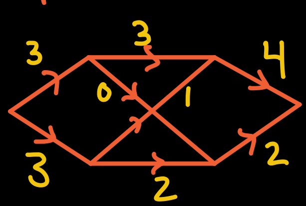
Idea: In flow-augmenting algorithm, pick s-t path in D_f of minimum length where "length" $\ell: A_f \cup B_f$ given by
$$\ell(a) = \begin{cases} k(a) & \text{if } a \in A_f \\ -k(a) & \text{if } a \in B_f. \end{cases}$$

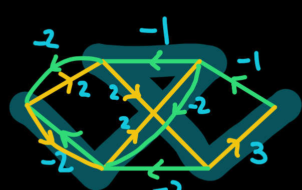
Ex: 

capacities



costs

3 steps $\rightsquigarrow$ 



$\rightsquigarrow$