

MA/AMA 514

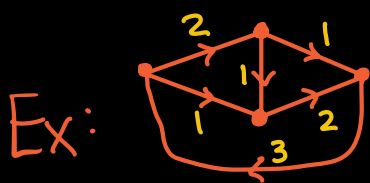
Today: §4.5, §4.6

§4.5 Circulations

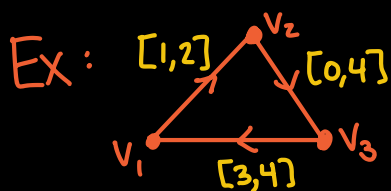
A function $f: A \rightarrow \mathbb{R}$ is a circulation on a directed graph $D=(V,E)$ if for every $v \in V$,

$$\sum_{a \in \delta^{\text{in}}(v)} f(a) = \sum_{a \in \delta^{\text{out}}(v)} f(a).$$

← flow conserved at every vertex!



Question: Given $d: A \rightarrow \mathbb{R}$, $c: A \rightarrow \mathbb{R}$, does $\exists$ a circulation f with $d(a) \leq f(a) \leq c(a) \forall a$?



no circulation possible!

Need $f(v_1 \rightarrow v_2) = f(v_3 \rightarrow v_1)$ for net flow 0 at v_1

But $f(v_1 \rightarrow v_2) \leq 2 < 3 \leq f(v_3 \rightarrow v_1)$

Hoffman's Circulation Thm Let $D=(V,A)$ be a directed graph and $d, c: A \rightarrow \mathbb{R}$ with $d(a) \leq c(a) \forall a \in A$.

Then $\exists$ a circulation f with $d(a) \leq f(a) \leq c(a) \forall a$

$$\iff \text{for all } U \subseteq V \quad \sum_{a \in \delta^{\text{in}}(U)} d(a) \leq \sum_{a \in \delta^{\text{out}}(U)} c(a).$$

(Proof) ($\Rightarrow$) For any circulation f and subset $U \subseteq V$,

$$\sum_{a \in \delta^{\text{in}}(U)} d(a) \leq \sum_{a \in \delta^{\text{in}}(U)} f(a) = \sum_{a \in \delta^{\text{out}}(U)} f(a) \leq \sum_{a \in \delta^{\text{out}}(U)} c(a).$$

($\Leftarrow$) For a function $f: A \rightarrow \mathbb{R}$, define

$$\text{loss}_f(v) = f(\delta^{\text{out}}(v)) - f(\delta^{\text{in}}(v)) \quad (\text{net loss of flow at } v)$$

Suppose $f: A \rightarrow \mathbb{R}$ minimizes $\|\text{loss}_f\|_1 := \sum_{v \in V} |\text{loss}_f(v)|$ over all functions f with $d \leq f \leq c$. Let

$$S = \{v \in V : \text{loss}_f(v) < 0\}, \quad T = \{v \in V : \text{loss}_f(v) > 0\}.$$

If $S = \emptyset$, then $T = \emptyset$ and f is a circulation.

Otherwise, define the directed graph $D_f = (V, A_f \cup B_f)$ where $A_f = \{a \in A : f(a) < c(a)\}$, $B_f = \{\bar{a} : a \in A, d(a) < f(a)\}$.

Claim: D_f does not have a path from S to T .

If $P: s \xrightarrow{a_1} v_1 \xrightarrow{a_2} \dots \xrightarrow{a_k} t$ is a path in D_f from $s \in S$ to $t \in T$,

define $f': A \rightarrow \mathbb{R}$ by $f'(a) = \begin{cases} f(a) + \alpha & \text{if } a \in A_f \cap P \\ f(a) - \alpha & \text{if } a \in B_f \cap P \\ f(a) & \text{otherwise.} \end{cases}$

where $\alpha > 0$ is sufficiently small.

Note: $\text{loss}_{f'}(v) = \text{loss}_f(v)$ for $v \neq s, t$, $\text{loss}_{f'}(s) = \text{loss}_f(s) + \alpha$, $\text{loss}_{f'}(t) = \text{loss}_f(t) - \alpha$

Then $d \leq f' \leq c$ and $\|\text{loss}(f')\|_1 < \|\text{loss}(f)\|_1 \not\equiv \square_{\text{claim}}$

Let U = set of vertices in D_f reachable from S .

For $a \in \delta^{\text{out}}(U)$, $a \notin A_f \Rightarrow f(a) = c(a)$

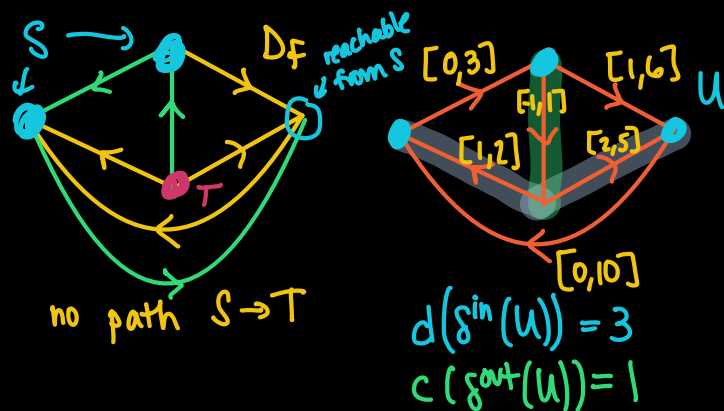
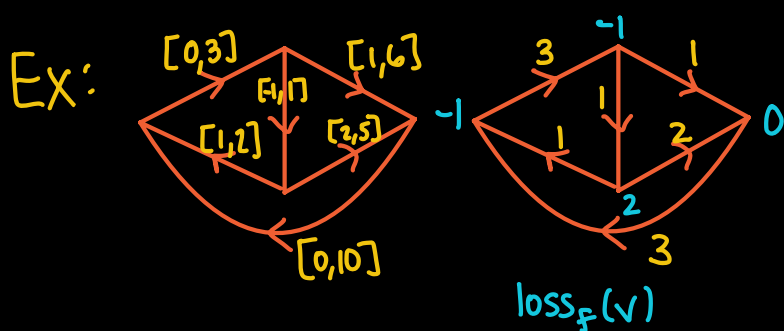
For $a \in \delta^{\text{in}}(U)$, $\bar{a} \notin B_f \Rightarrow f(a) = d(a)$.

$$\Rightarrow \sum_{a \in \delta^{\text{out}}(U)} c(a) - \sum_{a \in \delta^{\text{in}}(U)} d(a) = \sum_{a \in \delta^{\text{out}}(U)} f(a) - \sum_{a \in \delta^{\text{in}}(U)} f(a)$$

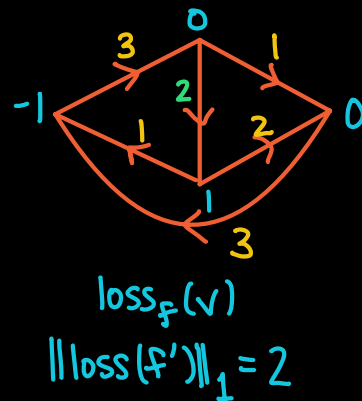
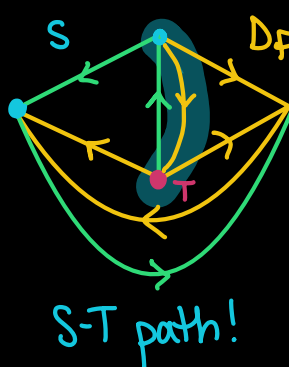
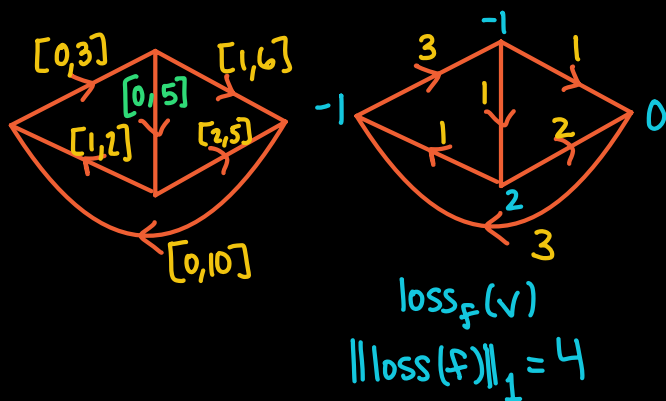
$$= \sum_{u \in U} \text{loss}_f(u) = \sum_{u \in S} \text{loss}_f(u) < 0.$$

By Claim $U \cap T = \emptyset \Rightarrow \text{loss}_f(u) = 0$ for $u \in U \setminus S$.

By Claim $U \cap T = \emptyset \Rightarrow \text{loss}_f(u) = 0$ for $u \in U \setminus S$. $\downarrow$



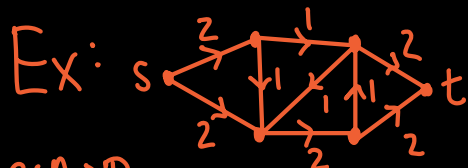
Ex: Augmenting path from claim?



§4.6 Minimum cost flows

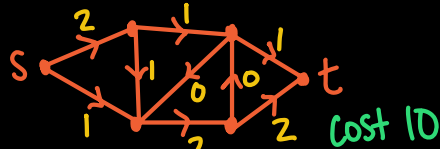
Given $D=(V,A)$, $s,t \in V$, capacities $c:A \rightarrow \mathbb{R}_{\geq 0}$ and costs $k:A \rightarrow \mathbb{R}_{\geq 0}$, find a flow $f:A \rightarrow \mathbb{R}$ of minimum cost among all flows under c of maximum value where

$$\text{cost}(f) = \sum_{a \in A} k(a) F(a)$$

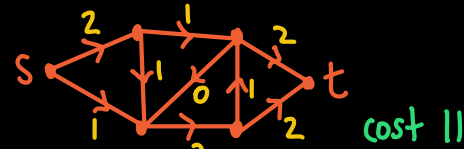


$C: A \rightarrow \mathbb{R}_{>0}$

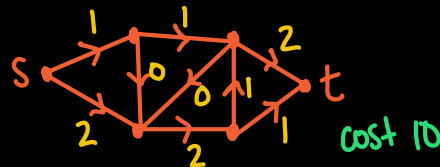
Max flows under C :



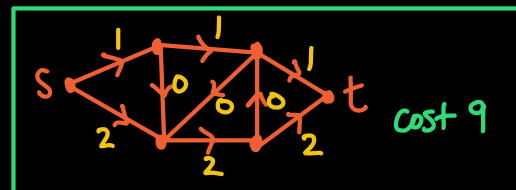
cost 10



cost 11



cost 10



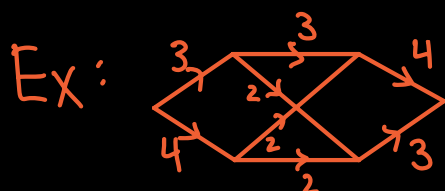
cost 9

min cost max flow $\uparrow$

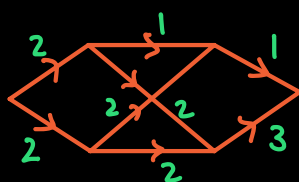
For details on how to find combinatorially, see §4.6.

Idea: In flow-augmenting algorithm, pick s - t path in D_f of minimum length where

"length" $l: A_f \cup B_f$ given by $l(a) = \begin{cases} k(a) & \text{if } a \in A_f \\ -k(a) & \text{if } a \in B_f. \end{cases}$



Capacities



costs

3 steps $\rightsquigarrow$

