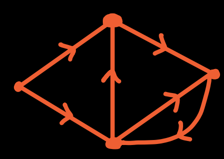
MA/AMA 514

Today §4.2 §4.3

§4.2 Flows in Networks

Let $D=(V,A)$ be a directed graph 

A = "arcs" $s \to t$ where $s,t \in V$.

A function $f: A \to \mathbb{R}_{\geq 0}$ is called an <u>s-t flow</u> for $s,t \in V$ if $\sum_{a \in \delta^{in}(v)} f(a) = \sum_{a \in \delta^{out}(v)} f(a)$ for all $v \in V \setminus \{s,t\}$

where $\delta^{in}(v) = \{\text{arcs in } A \text{ of the form } u \to v \text{ for } u \in V\}$

$\delta^{out}(v) = \{\text{arcs in } A \text{ of the form } v \to u \text{ for } u \in V\}$

"flow into v" = "flow out of v"

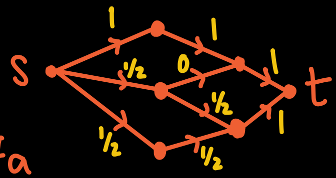
The <u>value</u> of a flow $f: A \to \mathbb{R}_{\geq 0}$ is

$$\text{value}(f) = \sum_{a \in \delta^{out}(s)} f(a) - \sum_{a \in \delta^{in}(s)} f(a)$$ = net flow leaving s (= net flow entering t)

Let $c: A \to \mathbb{R}_{>0}$ be a "capacity function".

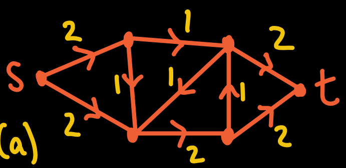
Say a flow $f: A \to \mathbb{R}_{\geq 0}$ is <u>under c</u> if $f(a) \leq c(a)$ $\forall a \in A$

<u>Max flow problem</u>: Given $D=(V,A)$, $s,t \in V$, $c: A \to \mathbb{R}_{>0}$, find an s-t flow f under c of maximum value.

Ex: 

$c(a)=1 \ \forall a$

flows $f: A \to \mathbb{R}_{\geq 0}$ under capacity c value = 2

Ex: 

$c(a)$

What is the max flow under c? Why?

For $U \subseteq V$, $\delta^{out}(U) = \{\text{arcs } u \to v \text{ with } u \in U, v \in V \setminus U\}$

Call U an <u>s-t cut</u> if $s \in U$ and $t \notin U$.

<u>Prop</u>: For any s-t flow f under c and any s-t cut $U \subseteq V$,

$$\text{value}(f) \leq c(\delta^{out}(U)) := \sum_{a \in \delta^{out}(U)} c(a).$$

(Proof)
$$\text{value}(f) = \sum_{a \in \delta^{out}(s)} f(a) - \sum_{a \in \delta^{in}(s)} f(a) + \overbrace{\sum_{u \in U \setminus \{s\}} \left(\sum_{a \in \delta^{out}(u)} f(a) - \sum_{a \in \delta^{in}(u)} f(a) \right)}^{=0}$$

$$= \sum_{u \in U} \left(\sum_{a \in \delta^{out}(u)} f(a) - \sum_{a \in \delta^{in}(u)} f(a) \right) = \sum_{a \in \delta^{out}(U)} f(a) - \sum_{a \in \delta^{in}(U)} f(a)$$

arcs within U cancel out

$$\leq \sum_{a \in \delta^{out}(U)} f(a) \leq \sum_{a \in \delta^{out}(U)} c(a) = c(\delta^{out}(U)).$$

For equality, both "$\leq$" must be "$=$"

<u>Max-Flow Min-Cut Thm</u>: For $D=(V,A)$, $s,t \in V$, $c: A \to \mathbb{R}_{\geq 0}$,

$$\max\{\text{value}(f) : f \text{ s-t flow under } c\} = \min\{c(\delta^{out}(U)) : U \text{ s-t cut}\}.$$

Proof by algorithm!

§4.3 Finding a max flow

Flow augmenting algorithm

<u>Input</u>: An s-t flow f

<u>Output</u>: either an s-t flow f' with $\mathrm{value}(f') > \mathrm{value}(f)$

or an s-t cut $U \subseteq V$ with $c(\delta^{out}(U)) = \mathrm{value}(f)$.

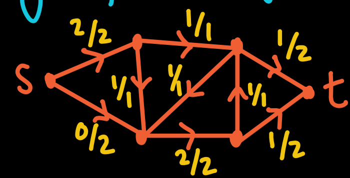
Cor: Max Flow = Min Cut !

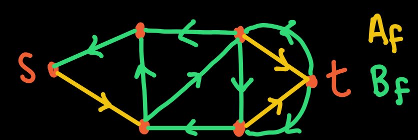
For arc $v \xrightarrow{a} w$ define a^{-1} to be $w \xrightarrow{a^{-1}} v$.

From flow f, make another digraph $D_f = (V, A_f \cup B_f)$ where

$A_f = \{a \in A : f(a) < c(a)\}$

$B_f = \{a^{-1} : a \in A,\ 0 < f(a)\}$.





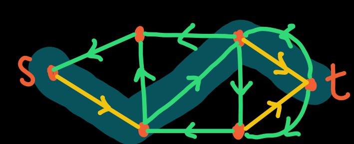
Case 1 ($\exists$ s-t path in D_f)

Let $P = (s = v_0 \xrightarrow{a_1} v_1 \xrightarrow{a_2} \cdots \to v_{k-1} \xrightarrow{a_k} v_k = t)$ be a simple path in D_f

Call P a <u>flow augmenting path</u>

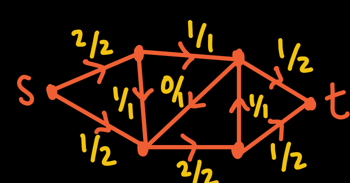
For $i = 1, \dots, k$, define $\sigma_i = \begin{cases} c(a_i) - f(a_i) & \text{if } a_i \in A_f \\ f(a_i^{-1}) - 0 & \text{if } a_i \in B_f \end{cases}$

Let $\alpha = \min\{\sigma_1, \dots, \sigma_k\}$ (note $\alpha > 0$)



Define $f' : A \to \mathbb{R}_{>0}$, as

$$f'(a) = \begin{cases} f(a) + \alpha & \text{if } a \in A_f \cap P \\ f(a) - \alpha & \text{if } a^{-1} \in B_f \cap P \\ f(a) & \text{otherwise} \end{cases}$$



Claim: f' is an s-t flow and $\text{value}(f') = \text{value}(f) + \alpha$

(1) $0 \le f'(a) \le c(a)$ for all $a \in A$

(By choice of α)

(2) f' is an s-t flow

Take $v \in V \setminus \{s,t\}$. If $v \notin P$, flow in/out of v unchanged.

$v \in P \implies a_i = (u_{i-1}, v),\ a_{i+1} = (v, u_{i+1})$ in P

Cases:

$a_i, a_{i+1} \in A_f$	$a_i \in A_f,\ a_{i+1} \in B_f$	$a_i \in B_f,\ a_{i+1} \in A_f$	$a_i, a_{i+1} \in B_f$
$+\alpha$ $+\alpha$	$+\alpha$ $-\alpha$	$-\alpha$ $+\alpha$	$-\alpha$ $-\alpha$
$\delta^{in}(v) \rightsquigarrow +\alpha$	$\delta^{in}(v) \to +\alpha - \alpha$	$\delta^{in}(v)$ same	$\delta^{in}(v) \rightsquigarrow -\alpha$
$\delta^{out}(v) \rightsquigarrow +\alpha$	$\delta^{out}(v)$ same	$\delta^{out}(v) \to -\alpha + \alpha$	$\delta^{out}(v) \rightsquigarrow -\alpha$

(3) $\text{value}(f') = \text{value}(f) + \alpha$

$a_1 = (s, v_1) \in P$ Cases:

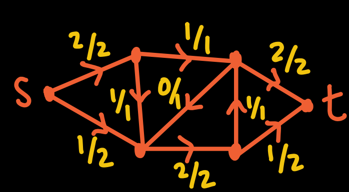
$a_1 \in A_f$ — s $+\alpha$

$\delta^{out}(s) \to +\alpha$ $\delta^{in}(s)$ same

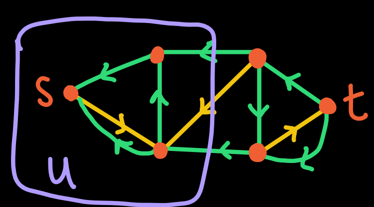
$a_1 \in B_f$ — s $-\alpha$

$\delta^{out}(s)$ same $\delta^{in}(s) \to -\alpha$

Case 2 ($\nexists$ s-t path in D_f)



Let $U = \{u \in V : \exists$ a path in D_f from s to $u\}$



Note: $s \in U$ and $t \notin U$. For $u \in U$, $v \notin U$

$u \xrightarrow{a} v \notin D_f \implies f(a) = c(a)$ and $f(a^{-1}) = 0$

$\implies c(\delta^{out}(U)) = \text{value}(f)$.