

MA/AMA 514

Today §4.2 §4.3

§4.2 Flows in Networks

Let $D=(V,A)$ be a directed graph



$A = \text{"arcs" } s \rightarrow t \text{ where } s, t \in V.$

A function $f: A \rightarrow \mathbb{R}_{\geq 0}$ is called an s-t flow for $s, t \in V$ if $\sum_{a \in \delta^{\text{in}}(v)} f(a) = \sum_{a \in \delta^{\text{out}}(v)} f(a)$ for all $v \in V \setminus \{s, t\}$

where $\delta^{\text{in}}(v) = \{\text{arcs in } A \text{ of the form } u \rightarrow v \text{ for } u \in V\}$
 $\delta^{\text{out}}(v) = \{\text{arcs in } A \text{ of the form } v \rightarrow u \text{ for } u \in V\}$

"flow into v " = "flow out of v "

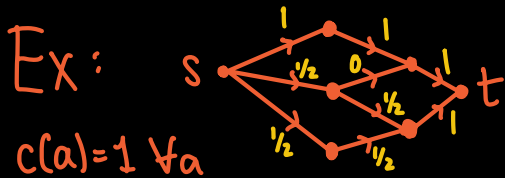
The value of a flow $f: A \rightarrow \mathbb{R}_{\geq 0}$ is

$$\text{value}(f) = \sum_{a \in \delta^{\text{out}}(s)} f(a) - \sum_{a \in \delta^{\text{in}}(s)} f(a) = \text{net flow leaving } s \quad (= \text{net flow entering } t)$$

Let $c: A \rightarrow \mathbb{R}_{\geq 0}$ be a "capacity function".

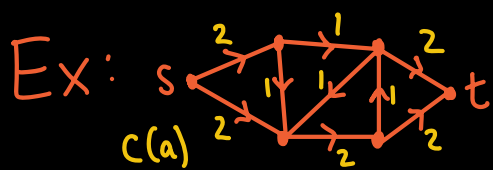
Say a flow $f: A \rightarrow \mathbb{R}_{\geq 0}$ is under c if $f(a) \leq c(a) \forall a \in A$

Max flow problem: Given $D=(V,A)$, $s, t \in V$, $c: A \rightarrow \mathbb{R}_{\geq 0}$, find an s-t flow f under c of maximum value.



flow $f: A \rightarrow \mathbb{R}_{\geq 0}$
under capacity c

value = 2



What is the max flow under c ? Why?

For $U \subseteq V$, $\delta^{\text{out}}(U) = \{\text{arcs } u \rightarrow v \text{ with } u \in U, v \in V \setminus U\}$



Call U an s-t cut if $s \in U$ and $t \notin U$.

Prop: For any s-t flow f under c and any s-t cut $U \subseteq V$,
 $\text{value}(f) \leq c(\delta^{\text{out}}(U)) := \sum_{a \in \delta^{\text{out}}(U)} c(a).$

$$\text{(Proof)} \quad \text{value}(f) = \sum_{a \in \delta^{\text{out}}(s)} f(a) - \sum_{a \in \delta^{\text{in}}(s)} f(a) + \overbrace{\sum_{u \in U \setminus \{s\}} \left(\sum_{a \in \delta^{\text{out}}(u)} f(a) - \sum_{a \in \delta^{\text{in}}(u)} f(a) \right)}^{=0}$$

$$= \sum_{u \in U} \left(\sum_{a \in \delta^{\text{out}}(u)} f(a) - \sum_{a \in \delta^{\text{in}}(u)} f(a) \right) = \sum_{a \in \delta^{\text{out}}(U)} f(a) - \sum_{a \in \delta^{\text{in}}(U)} f(a)$$

arcs within U cancel out


$$\leq \sum_{a \in \delta^{\text{out}}(U)} f(a) \leq \sum_{a \in \delta^{\text{out}}(U)} c(a) = c(\delta^{\text{out}}(U)).$$

For equality, both
"≤" must be "="

Max-Flow Min-Cut Thm: For $D=(V,A)$, $s,t \in V$, $c: A \rightarrow \mathbb{R}_{\geq 0}$,
 $\max \{ \text{value}(f) : f \text{ s-t flow under } c \}$
 $= \min \{ c(\delta^{\text{out}}(U)) : U \text{ s-t cut} \}.$

Proof by algorithm!

§4.3 Finding a max flow

Flow augmenting algorithm

Input: An s-t flow f

Output: either an s-t flow f' with $\text{value}(f') > \text{value}(f)$
or an s-t cut $U \subseteq V$ with $c(\delta^{\text{out}}(U)) = \text{value}(f)$.

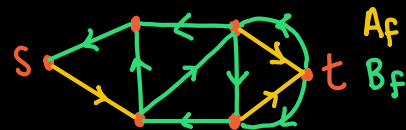
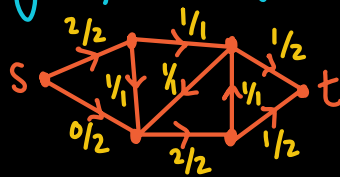
Cor: Max Flow = Min Cut !

For arc $v \xrightarrow{a} w$ define $\bar{a}'$ to be $w \xrightarrow{\bar{a}'} v$.

From flow f , make another digraph $D_f = (V, A_f \cup B_f)$ where

$$A_f = \{a \in A : f(a) < c(a)\}$$

$$B_f = \{\bar{a}' : a \in A, 0 < f(a)\}$$



Case 1 ($\exists$ s-t path in D_f)

Let $P = (s = v_0 \xrightarrow{a_1} v_1 \xrightarrow{a_2} \dots \xrightarrow{a_k} v_k = t)$ be a simple path in D_f

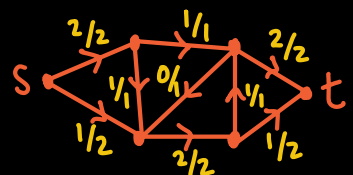
Call P a flow augmenting path

For $i=1, \dots, k$, define $\sigma_i = \begin{cases} c(a_i) - f(a_i) & \text{if } a_i \in A_f \\ f(a_i') - 0 & \text{if } a_i \in B_f \end{cases}$

Let $\alpha = \min\{\sigma_1, \dots, \sigma_k\}$ (note $\alpha > 0$)

Define $f' : A \rightarrow \mathbb{R}_{>0}$, as

$$f'(a) = \begin{cases} f(a) + \alpha & \text{if } a \in A_f \cap P \\ f(a) - \alpha & \text{if } \bar{a}' \in B_f \cap P \\ f(a) & \text{otherwise} \end{cases}$$



Claim: f' is an s-t flow and $\text{value}(f') = \text{value}(f) + \alpha$

(1) $0 \leq f'(a) \leq c(a)$ for all $a \in A$

(By choice of α)

(2) f' is an s-t flow

Take $v \in V \setminus \{s, t\}$. If $v \notin P$, flow in/out of v unchanged.

$v \in P \Rightarrow a_i = (u_{i-1}, v), a_{i+1} = (v, u_{i+1})$ in P

Cases:

$a_i, a_{i+1} \in A_f$



$\delta^{\text{in}}(v) \rightarrow +\alpha$

$\delta^{\text{out}}(v) \rightarrow +\alpha$

$a_i \in A_f, a_{i+1} \in B_f$



$\delta^{\text{in}}(v) \rightarrow +\alpha - \alpha$

$\delta^{\text{out}}(v)$ same

$a_i \in B_f, a_{i+1} \in A_f$



$\delta^{\text{in}}(v)$ same

$\delta^{\text{out}}(v) \rightarrow -\alpha + \alpha$

$a_i, a_{i+1} \in B_f$



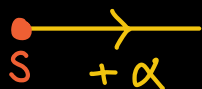
$\delta^{\text{in}}(v) \rightarrow -\alpha$

$\delta^{\text{out}}(v) \rightarrow -\alpha$

(3) $\text{value}(f') = \text{value}(f) + \alpha$

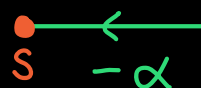
$a_i = (s, v_1) \in P$ Cases:

$a_i \in A_f$



$\delta^{\text{out}}(s) \rightarrow +\alpha$ $\delta^{\text{in}}(s)$ same

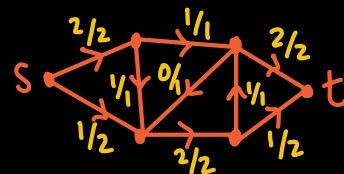
$a_i \in B_f$



$\delta^{\text{out}}(s)$ same

$\delta^{\text{in}}(s) \rightarrow -\alpha$

Case 2 ($\nexists$ s-t path in D_f)

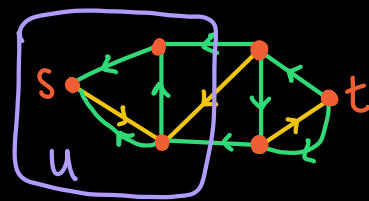


Let $U = \{u \in V : \exists \text{ a path in } D_f \text{ from } s \text{ to } u\}$

Note: $s \in U$ and $t \notin U$. For $u \in U, v \notin U$

$u \xrightarrow{a} v \notin D_f \Rightarrow f(a) = c(a)$ and $f(a^{-1}) = 0$

$\Rightarrow c(\delta^{\text{out}}(U)) = \text{value}(f)$



full cap out
nothing in