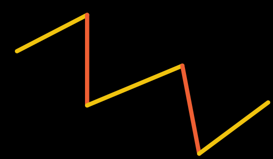
MA/AMA 514

Today: Finding max matchings (§3.4)
and matching polytopes (§3.6)

§3.4 Algorithm for max. matchings in bipartite graphs

Last time:

1) If a matching M is not max sized then $\exists$ an M-augmenting path

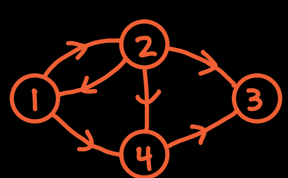


2) For bipartite graphs, $\nu(G) = \tau(G)$
that is $|\text{max matching}| = |\text{min vertex cover}|$

Challenge: Finding M-augmenting paths
We'll use directed graphs to do this

A directed graph is a pair $D = (V, A)$
V = "vertices" (finite set)
A = "arcs" = ordered pairs of vertices (v,w)

Ex:



$V = \{1,2,3,4\}$ $A = \{(1,2),(2,1),(1,4),(2,4),(2,3),(4,3)\}$

A path in P consists of arcs $(v_0,v_1),(v_1,v_2),\ldots,(v_{k-1},v_k)$ with $v_i \neq v_j$ for $i \neq j$. Paths (and shortest paths) between $v, w \in V$ can be computed in poly time. (See §1.1)

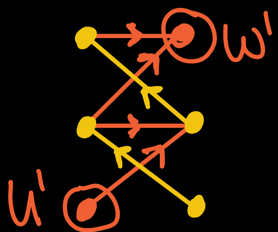
Algorithm:

Input: Bipartite graph $G=(V,E)$, matching M

Output: Larger matching M' (if one exists)

Write $V = U \cup W$.

For $e=\{u,w\}\in E$ with $u\in U$, $w\in W$, direct edge e $w \to u$ if $e\in M$ and $u\to w$ if $e\notin M$.



Let U', W' denote the set of vertices of U, W (resp.) uncovered by M

Claim: A path $P \subseteq E$ is M-augmenting iff it gives a directed path from some $u'\in U'$ to some $w'\in W'$.

(Proof) If P is M-augmenting, its endpoints lie in $U' \cup W'$.

Length(P) odd $\Rightarrow$ one end pt lies in U', one in W'

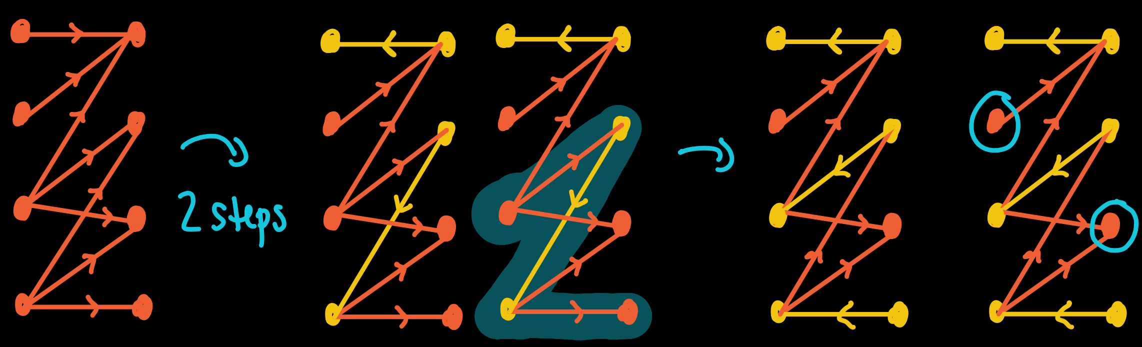
Alternating edges lie in M $\Rightarrow$ all edges directed $u' \to w'$

Conversely, if P is a directed path from $u'\in U'$ to $w'\in W'$ then length(P) is odd and because of edge directions edges must alternate between M and $E\setminus M$.

One can find such a directed path P in $O(|E|)$ time (if it exists). Take $M' = P \Delta M$ (a bigger matching!)

Starting from $M=\emptyset$, repeat $O(|V|)$ times to get max matching.

Improved alg. by Hopcraft, Karp (1973) has running time $O(|V|^{1/2}|E|)$.



no path $u' \to w'$
$\Longrightarrow$ no M-aug. path
$\Longrightarrow$ M = max. matching

Application: Assignment problem

There are k machines $m_1, \dots, m_k$ and n jobs $j_1, \dots, j_n$. Each machine is capable of doing some of the jobs but can only do one job at a time.

Goal: assign jobs to machines to do the largest # of jobs at the same time.

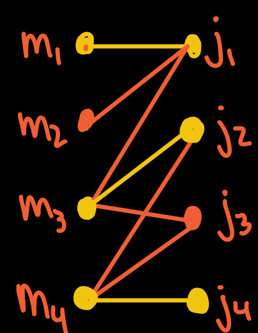
Create a bipartite graph with vertices $\{m_1, \dots, m_k, j_1, \dots, j_n\}$ and edges $\{m_i, j_\ell\}$ if machine m_i can do job j_ℓ.

Assignment maximizing # job $\Longleftrightarrow$ max. matching

Ex:

	j_1	j_2	j_3	j_4
m_1	X			
m_2	X			
m_3	X	X	X	
m_4		X	X	X

$\longleftrightarrow$

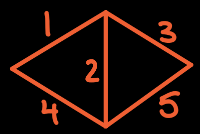


See §3.5 for max. weighted matchings

§3.6 The matching polytope

Given a graph $G=(V,E)$ and matching M of G, define $\mathbb{1}_M \in \mathbb{R}^E$ by $\mathbb{1}_M(e) = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{if } e \notin M \end{cases}$

Ex:



$\{3,4\} \longleftrightarrow (0,0,1,1,0)$

$\{2\} \longleftrightarrow (0,1,0,0,0)$

$\emptyset \longleftrightarrow (0,0,0,0,0)$

Two polytopes:

The matching polytope of a graph $G=(V,E)$ is

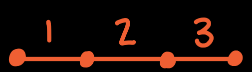
$$P_{\text{matching}}(G) = \operatorname{conv}(\{\mathbb{1}_M : M \text{ is a matching in } G\})$$

and the perfect matching polytope is

$$P_{\text{perfect matching}}(G) = \operatorname{conv}(\{\mathbb{1}_M : M \text{ is a perfect matching of } G\})$$

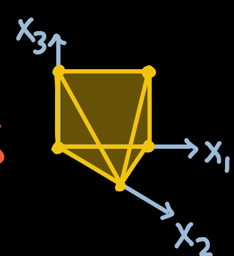
could be empty!

Ex:



$P_{\text{matching}} = \operatorname{conv}\{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,0,1)\}$

$P_{\text{perfect matching}}(G) = \{(1,0,1)\}$



Question: What are the linear ineq. defining $P_{\text{matching}}(G)$?

If $x=(x_e)_{e\in E}$ belongs to $P_{\text{matching}}(G)$, then

1) $0 \le x_e \le 1$ for all $e \in E$
2) $\sum_{e \ni v} x_e \le 1$ for all $v \in V$ (=1 for perfect matchings)

Define $Q_{\text{matching}}(G) = \left\{ x \in \mathbb{R}^E : \begin{array}{l} x_e \geq 0 \ \forall e \in E \\ \sum_{e \ni v} x_e \leq 1 \ \forall v \in V \end{array} \right\}$

$Q_{\text{perfect matching}}(G) = \left\{ x \in \mathbb{R}^E : \begin{array}{l} x_e \geq 0 \ \forall e \in E \\ \sum_{e \ni v} x_e = 1 \ \forall v \in V \end{array} \right\}$

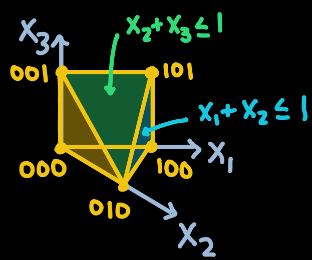
The polytopes P_*, Q_* have the same integer points.

When are they the same polytope?

Ex: $Q_{\text{matching}}(\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet) = \{x \in \mathbb{R}^3_{>0} : x_1 \leq 1,\ x_1 + x_2 \leq 1,\ x_2 + x_3 \leq 1,\ x_3 \leq 1\}$

$Q_{\text{perfect matching}}(\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet) = \{x \in \mathbb{R}^3_{>0} : x_1 = 1,\ x_1 + x_2 = 1,\ x_2 + x_3 = 1,\ x_3 = 1\}$

In this case $P_* = Q_*$!



In Hwk 3, you'll show that this doesn't hold for all graphs!

Thm: If G is bipartite then $P_{\text{matching}}(G) = Q_{\text{matching}}(G)$ and $P_{\text{perfect matching}}(G) = Q_{\text{perfect matching}}(G)$.

$\Rightarrow$ We can find max matching by solving the LP

$$\max\{\mathbb{1}^T x : x_e \geq 0 \ \forall e \in E,\ \sum_{e \ni v} x_e \leq 1 \ \forall v \in V\}$$