MA/AMA 514

Today: Matchings & bipartite graphs (§3.2, 3.3)

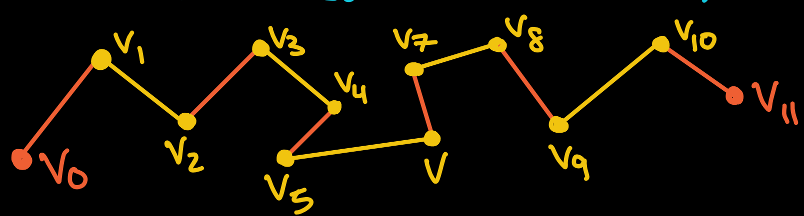
§3.2 M-augmenting paths

Recall: A <u>matching</u> of a graph $G=(V,E)$ is a subset $M \subseteq E$ s.t. $e \cap e' = \emptyset$ for all $e \neq e' \in M$.

Let $\nu(G) = \max\{|M| : M \text{ is a matching in } G\}$.

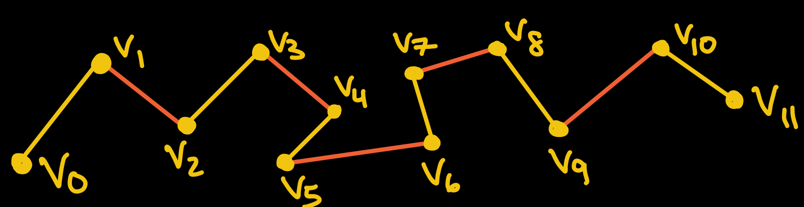
<u>Def</u>: Let M be a matching in $G=(V,E)$.
A path $P=(v_0, \dots, v_{2t+1})$ in G is <u>M-augmenting</u> if

$v_1v_2, v_3v_4, \dots, v_{2t-1}v_{2t} \in M$ and $v_0, v_{2t+1} \notin \bigcup_{e \in M} e$.



Then $M' = M \,\Delta\, E(P)$ is a matching in G.

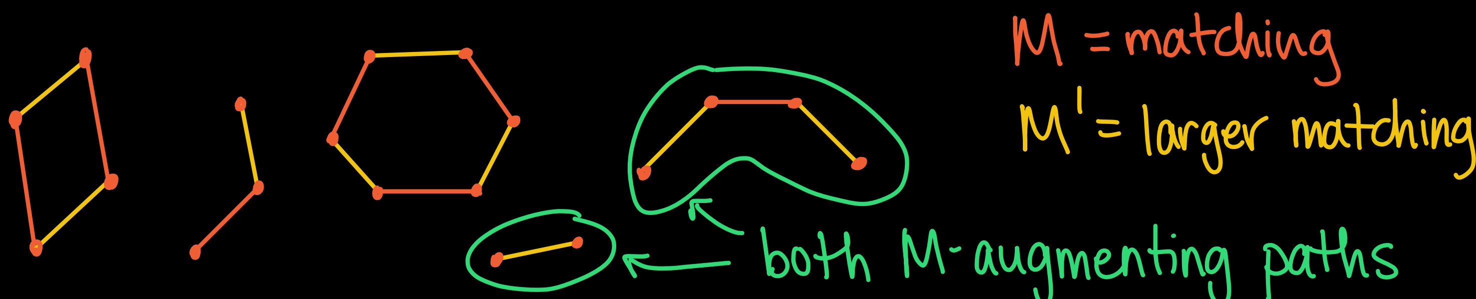
$(= M \setminus \{v_1v_2, \dots, v_{2t-1}v_{2t}\} \cup \{v_0v_1, v_2v_3, \dots, v_{2t}v_{2t+1}\})$



<u>Thm 3.2</u> If M is a matching in G, either $|M| = \nu(G)$ or there exists an M-augmenting path.

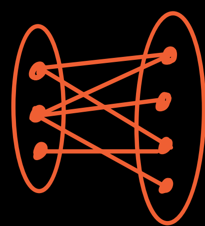
(Proof) If there is an M-augmenting path, then M' is a matching with $|M'| > |M|$, so $|M|$ not maximal.

Conversely, suppose M' is a matching with $|M'| > |M|$.
Consider $G' = (V, M \triangle M')$. Every $v \in V$ has degree ≤ 2 in G'!
$\Rightarrow$ each connected component of G' is a path or a circuit.
Since $|M'| > |M|$, some connected component of G' has more edges from M' than from M.
This component must be an M-augmenting path!



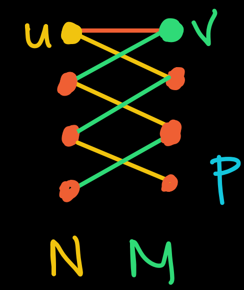
§3.3 König's Theorems

A graph $G = (V, E)$ is <u>bipartite</u> if there is a partition of the vertex set $V = V_1 \uplus V_2$ with no edges between vertices in the same part.
That is, $|e \cap V_i| = 1$ for all $e \in E$, $i = 1, 2$.

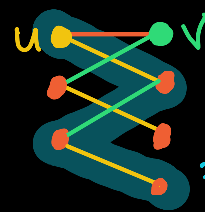


<u>Lemma:</u> If G is bipartite with edge $e = \{u, v\}$ then either u or v is covered by EVERY max. matching.
$\Rightarrow$ there is a vertex covered by every matching of largest size!

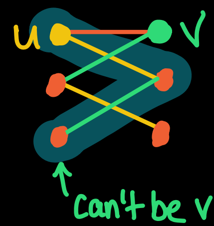
(Proof) Let $e = \{u, v\} \in E$ and suppose there exist matchings M, N of maximal size where M doesn't cover u and N doesn't cover v.



Since neither $M \cup e$ nor $N \cup e$ are matchings M covers v and N covers u.



Let P be the connected component of $M \cup N$ containing u. Since u is not covered by M, u has deg 1 in $M \cup N$ and P is a path ending in u. If length(P) is odd, then P is a M-augmenting path $\Rightarrow$ M not a max. matching $\divideontimes$



If length(P) is even, then P does not end at v since any path starting in V_1 and ending in V_2 has an odd number of edges. The deg of v in $M \cup N$ also equals 1 and so P cannot cover v.
It follows that $P \cup e$ is an N-augmenting path $\divideontimes$.

Thm 3.3 (König's Thm) For a bipartite graph G,

matching # $= \nu(G) = \tau(G) =$ vertex cover #

Last time: showed $\nu(G) \leq \tau(G)$ for arbitrary graphs G

and $\nu(K_3) = 1 < 2 = \tau(K_3)$ (note K_3 <u>not</u> bipartite!)

(Proof) By induction on $|E|$.

If $E=\emptyset$, then $\nu(G)=0=\tau(G)$. So assume $E\neq\emptyset$.

By Lemma, some vertex $u\in e$ is containg in every every largest matching of G.

Let $G'=G-u$ (remove u and all edges containing it)

Then $\nu(G')=\nu(G)-1$. Why?

M largest matching in $G \Rightarrow \exists e\in M$ with $u\in e$

$\Rightarrow M\setminus e$ matching of $G' \Rightarrow \nu(G')\geq \nu(G)-1$.

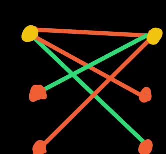
Any matching M' of G' is also a matching of G that doesn't cover $u \Rightarrow M'$ not a max. matching of G

$\Rightarrow |M'|\leq \nu(G)-1$.

By induction, $\nu(G')=\tau(G')$. Let $C\subseteq V\setminus\{u\}$ be a vertex cover of G' with size $\tau(G')$. Every edge $e\in E$ either contains u or is an edge in $G' \Rightarrow$ contains some $v\in C$.

$\Rightarrow C\cup\{u\}$ is a vertex cover of G

$\Rightarrow \tau(G)\leq |C\cup\{u\}| = \nu(G')+1=\nu(G)$.



Cor: If G is bipartite and has no isolated vertices, stable set # $=\alpha(G)=\rho(G)=$ edge cover #

Proof by Gallai's Thm: $\alpha(G)+\tau(G)=\nu(G)+\rho(G) \Rightarrow \tau(G)-\nu(G)=\rho(G)-\alpha(G)$