MA/AMA 514

Today Matchings and covers (§3.1)

Linear programming for combinatorial optimization

Combinatorial optimization problems:

Given $\omega : [m] \to \mathbb{R}$ and $\mathcal{S}$ = collection of subsets of $[m]$,

find $\max_{S \in \mathcal{S}} \sum_{i \in S} \omega(i)$

ex: G = graph with m edges, $\mathcal{S}$ = {spanning trees of G}

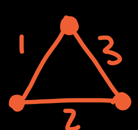
For $S \subseteq [m]$, let $\mathbb{1}_S \in \mathbb{R}^m$ denote the indicator vector of S:

$$\mathbb{1}_S(i) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$$

Idea: Let $P = \mathrm{conv}(\{\mathbb{1}_S : S \in \mathcal{S}\})$. Then

$$\max\{\sum_{i \in S} \omega(i) : S \in \mathcal{S}\} = \max\{\omega^T x : x \in P\}$$ ← a linear program!

(know max acheived by some $\mathbb{1}_S$)

Ex: G = 

$\mathcal{S} = \{\{1,2\},\{1,3\},\{2,3\}\} \to P = \mathrm{conv}(\{(1,1,0),(1,0,1),(0,1,1)\})$

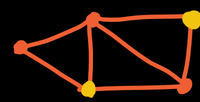
Question: For what combinatorial problems (i.e. sets $\mathcal{S}$) does P have a polynomial size description as $P = \{x \in \mathbb{R}^m : Ax \le b\}$?

What is the dual LP?

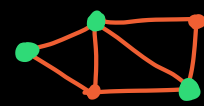
Matchings & covers graphs

Many definitions: Let $G=(V,E)$ be a graph.

1) A subset $S \subseteq V$ of vertices is <u>stable</u> (or <u>independent</u>) it contains no edges ($\forall i,j \in S$, $\{i,j\} \notin E$)



2) A subset $W \subseteq V$ is a <u>vertex cover</u>



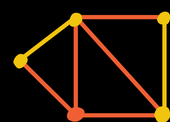
if for all edges $e \in E$, $e \cap W$ is nonempty.

Remarks: • $S \subseteq V$ stable, $S' \subseteq S \Rightarrow S'$ stable

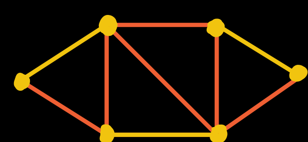
• $W \subseteq V$ is a vertex cover, $W' \supseteq W \Rightarrow W'$ is a vertex cover

• $S \subseteq V$ stable $\Longleftrightarrow V \setminus S$ is a vertex cover (You check!)

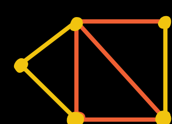
3) A subset $M \subseteq E$ of edges is a <u>matching</u> if no edges share a vertex ($e \neq e' \in M \Rightarrow e \cap e' = \phi$)



A matching is <u>perfect</u> if it covers all vertices: $\bigcup_{e \in M} e = V \implies |M| = \frac{1}{2}|V|$



4) A subset $F \subseteq E$ is an <u>edge cover</u> if it covers all vertices (ie. $\bigcup_{e \in F} e = V$).



(Edge covers only exist if G has no isolated vertices)

Remarks: • $M \subseteq E$ a matching, $M' \subseteq M \Rightarrow M'$ a matching

• $F \subseteq E$ an edge cover, $F' \supseteq F \Rightarrow F'$ an edge cover

Define: $\alpha(G) = \max\{|S| : S \subseteq V \text{ stable}\}$ "stable set #"

$\tau(G) = \min\{|W| : W \text{ vertex cover}\}$ "vertex cover #"

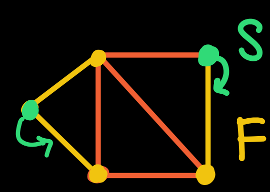
$\nu(G) = \max\{|M| : M \text{ matching}\}$ "matching #"

$\rho(G) = \min\{|F| : F \text{ edge cover}\}$ "edge cover #"

Claim: 1) $\alpha(G) \leq \rho(G)$ and 2) $\nu(G) \leq \tau(G)$

(Proof) Let $S \subseteq V$ be stable and $F \subseteq E$ be an edge cover.

Each vertex $v \in S$ is contained in some $e_v \in F$



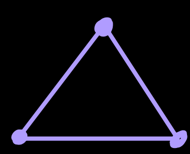
Since S stable, no two vertices $v, v' \in S$ are contained in the same edge $e \in F$.

$\Rightarrow$ We can define an injective map $S \to F \Rightarrow |S| \leq |F|$.

$\Rightarrow \alpha(G) = \max\{|S| : S \text{ stable}\} \leq \min\{|F| : F \text{ edge cover}\} = \rho(G)$

2) Similar!

Note: These inequalities can be strict!



Ex: K_3 $\quad \alpha(G) = 1 < 2 = \rho(G) \qquad \tau(G) = 2 > \nu(G) = 1$

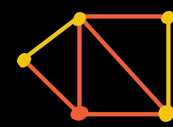
<u>Thm 3.1 (Gallai's Thm)</u> For any graph $G = (V, E)$ with no isolated vertices

$$\alpha(G) + \tau(G) = |V| = \nu(G) + \rho(G).$$

<u>Cor</u>: $\alpha(G) = \rho(G) \Leftrightarrow \nu(G) = \tau(G)$

(Proof) $\alpha(G)+\tau(G)=|V|$ since S stable $\Leftrightarrow V\setminus S$ vertex cover

Remains to show $\nu(G)+\rho(G)=|V|$.

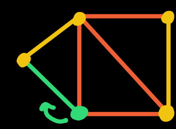


unmatched

Take a matching M of size $|M|=\nu(G)$.

There are $|V|-2|M|$ unmatched vertices.

For each, add an edge to M covering it.

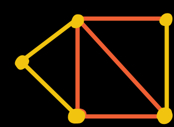


edge cover

This gives an edge cover F with

$$|F| \leq |M| + (|V|-2|M|) = |V|-|M|$$

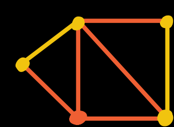
$$\Rightarrow \rho(G) \leq |F| \leq |V| - \nu(G) \quad \Rightarrow \quad \nu(G)+\rho(G) \leq |V|$$



Let F be an edge cover of size $\rho(G)$

For each $v \in V$, delete from F all but one edge incident to v.

The result is a matching M of G.



If $\deg_F(v)$ denotes $\#\{e \in F : v \in e\}$, then

$$|M| \geq |F| - \sum_{v\in V} (\deg_F(v)-1) = |F| - \underbrace{\sum_{v \in V} \deg_F(v)}_{=2|F|} + |V|$$

every edge $e \in F$ contributes 2

$$\Rightarrow |M| \geq |F| - 2|F| + |V| = |V| - |F| = |V| - \rho(G).$$

$$\Rightarrow \nu(G) \geq |M| \geq |V| - \rho(G) \quad \Rightarrow \quad \nu(G)+\rho(G) \geq |V|. \quad \square$$

Proof shows that to get a smallest edge cover F from a largest matching M, add one edge covering each missing vertex (can't be any edge between vertices uncovered by M).

Next time: how can we solve these optimization problem?

Greedy algorithm does not always work!

Ex: max matching in  ?

 matching M, $|M| = 2$

We can't add any more edges, but M is not optimal!

 matching $\tilde{M}$, $|\tilde{M}| = 3$

Such paths will be called "M-augmenting".