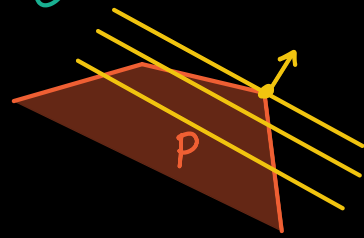
# MA/AMA 514

Today: Duality in Linear Programming (§2.3, 2.4)

(LP) $\max\{c^T x : Ax \le b\}$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

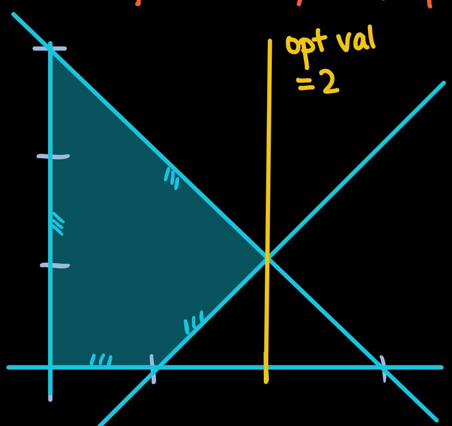



Polyhedron $P = \{x : Ax \le b\} \subseteq \mathbb{R}^n$, defined by m inequalities

## Duality in Linear Programming

Idea: to a linear program (P) (for "primal")
we associate a dual linear program (D) that aims
to find the best possible upper bound on (P)

Ex 1: $\max\ x_1$ s.t. $x_1 \ge 0$, $x_2 \ge 0$, $x_1 + x_2 \le 3$, $x_1 - x_2 \le 1$



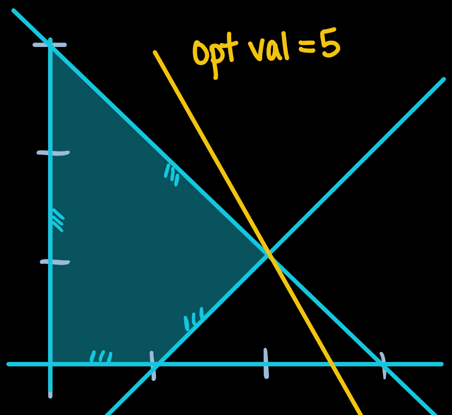

Can we see $x_1 \le 2$ from inequalities?

Yes: $x_1 + x_2 \le 3 \overset{}{\underset{\times 1/2}{\Longrightarrow}} \frac{1}{2}x_1 + \frac{1}{2}x_2 \le \frac{3}{2}$

$x_1 - x_2 \le 1 \underset{\times 1/2}{\Longrightarrow} \frac{1}{2}x_1 - \frac{1}{2}x_2 \le \frac{1}{2}$

sum: $x_1 \le 2$

Ex 2: $\max\ 2x_1 + x_2$ s.t. $x_1 \ge 0$, $x_2 \ge 0$, $x_1 + x_2 \le 3$, $x_1 - x_2 \le 1$




Can we see $2x_1 + x_2 \le 5$ from inequalities?

Yes: $x_1 + x_2 \le 3 \underset{\times 3/2}{\Longrightarrow} \frac{3}{2}x_1 + \frac{3}{2}x_2 \le \frac{9}{2}$

$x_1 - x_2 \le 1 \underset{\times 1/2}{\Longrightarrow} \frac{1}{2}x_1 - \frac{1}{2}x_2 \le \frac{1}{2}$

sum: $2x_1 + x_2 \le 5$

What inequalities $c^T x \le c_0$ hold on $P = \{x \in \mathbb{R}^n : Ax \le b\}$?

Know inequalities $a_j^T x \le b_j$ hold on $P$

$\Rightarrow$ for any $y_1 \ge 0, \dots, y_m \ge 0$, $\left.\begin{array}{c} y_1 a_1^T x \le y_1 b_1 \\ \vdots \\ y_m a_m^T x \le y_m b_m \end{array}\right\}$ all hold on $P$

$$\Rightarrow \quad y^T A x = \left( \sum_{j=1}^m y_j a_j^T \right) x \le \left( \sum_{j=1}^m y_j b_j \right) = y^T b$$

when $y_1 \ge 0, \dots, y_m \ge 0$, this inequality "obviously" holds on $P$

If $y^T A$ equals cost vector $c^T$, this gives an upper bound (of $y^T b$) on

$$(P) \quad \max\ c^T x \quad \text{s.t.} \quad Ax \le b.$$

In Ex 1, taking $y^T = (1/2, 1/2, 1, 1/2)$ gives

$$\begin{array}{rl} 1/2 \times (-x_1 & \le 0) \\ 1/2 \times (\quad -x_2 & \le 0) \\ 1 \times (x_1 + x_2 & \le 3) \\ 1/2 \times (x_1 - x_2 & \le 1) \\ \hline x_1 & \le 7/2 \end{array}$$

This shows that $x_1 \le 7/2$ is a valid inequality on $\{x \in \mathbb{R}^2 : x_1 \ge 0, x_2 \ge 0, x_1 + x_2 \le 3, x_1 - x_2 \le 1\}$.

We saw above that an even better bound of $x_1 \le 2$ holds.

The <u>dual linear program</u> is

(D) $\quad \min b^T y$ s.t. $A^T y = c$ and $y \geq 0$.

This looks for the best "obvious" upper bound on (P).

<u>Prop (Weak Duality)</u> If $x$ is feasible for (P) and $y$ is feasible for (D), then

$$c^T x \leq b^T y.$$

(Proof) $x$ feasible for (P) $\iff Ax \leq b$

$y$ feasible for (D) $\iff A^T y = c,\ y \geq 0$

If both hold,

$$c^T x = (A^T y)^T x = y^T A x \underset{\substack{\uparrow \\ \text{uses } y \geq 0}}{\leq} y^T b = b^T y$$

Cor: If $x^*$ is feasible for (P), $y^*$ is feasible for (D), and $c^T x^* = b^T y^*$ then both are optimal for (P), (D) resp.

Ex 1: (P) $\max x_1$ s.t. $\underbrace{\overset{y_1}{-x_1 \leq 0},\ \overset{y_2}{-x_2 \leq 0},\ \overset{y_3}{x_1 + x_2 \leq 3},\ \overset{y_4}{x_1 - x_2 \leq 1}}$

$Ax \leq b$ for $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \end{pmatrix}$

(D) $\min 3y_3 + y_4$ s.t. $\underbrace{\begin{array}{l} -y_1 + y_3 + y_4 = 1 \\ -y_2 + y_3 - y_4 = 0, \end{array}}$ $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \geq 0$

$A^T y = c \rightarrow$

$x^* = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ feasible for (P) $\qquad x_1^* = 2 = 3y_3^* + y_4^*$

$y^* = \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{pmatrix}$ feasible for (D) $\qquad \Rightarrow$ x opt. sol. for (P) and y opt sol. for (D)

<u>Thm 2.6</u> (Strong Duality)

$$\max\{c^T x : Ax \le b\} = \min\{y^T b : y \ge 0,\ y^T A = c^T\}$$

provided that both sets are nonempty.

Why? Main ingredients:

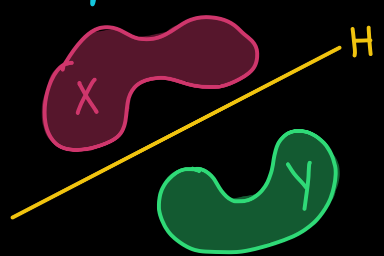
1) Separation thm for convex sets

A hyperplane H <u>separates</u> sets $X, Y \subseteq \mathbb{R}^n$ if they ie in different connected components of $\mathbb{R}^n \setminus H$

$H = \{x \in \mathbb{R}^n : a^T x = b\}$

$X \subseteq \{x \in \mathbb{R}^n : a^T x < b\}$

$Y \subseteq \{x \in \mathbb{R}^n : a^T x > b\}$




<u>Thm 2.1</u> Let $C \subseteq \mathbb{R}^n$ be a closed convex set. For any point $z \in \mathbb{R}^n$ with $z \notin C$, there is a hyperplane H separating z from C. (See notes for proof)

2) Farkas' Lemma

There is a nonnegative vector x with $Ax = b$ iff there is no solution $y \in \mathbb{R}^m$ to $y^T A \ge 0$ and $y^T b < 0$, i.e.

$$\exists x \in \mathbb{R}^n_{\ge 0} \text{ s.t. } Ax = b \iff \nexists y \in \mathbb{R}^m \text{ s.t. } y^T A \in \mathbb{R}^n_{\ge 0},\ y^T b < 0.$$

(Proof) ($\Rightarrow$) Suppose $\exists x \in \mathbb{R}^n_{\geq 0}$ with $Ax=b$.

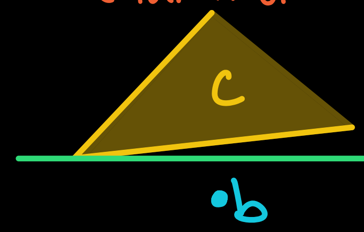
If $y \in \mathbb{R}^m$ and $y^TA \in \mathbb{R}^n_{\geq 0}$, then $(y^TA)x = \sum_{i=1}^n (y^TA)_i x_i \geq 0$.

$\Rightarrow 0 \leq y^T(Ax) = y^Tb$.

($\Leftarrow$) (By contrapositive) Suppose there is no $x \in \mathbb{R}^n_{\geq 0}$ s.t. $Ax=b$.

Let $C = \{Ax : x \in \mathbb{R}^n_{\geq 0}\} = \text{cone}(A^1, \dots, A^n)$ where $A^i = i^{th}$ column of $A$

By assumption $b \notin C \Rightarrow$ there exists a hyperplane $H = \{z : c^Tz = c_0\}$ separating $b$ from $C$.




Translating $H$ to touch $C$, we can take $c_0 = 0$. *

Then $c^Tz \geq 0$ for all $z \in C \Rightarrow c^TA_i \geq 0$ for $i = 1, \dots, n$

$\Rightarrow c^TA \in \mathbb{R}^n_{\geq 0}$

and $c^Tb < 0$. Take $y = c$.

* Formally, $c^Tz > c_0$ for $z \in C$ and $c^Tb < c_0$.

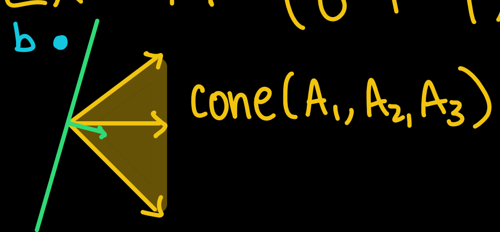
$0 \in C \Rightarrow c^T0 = 0 > c_0 \Rightarrow c^Tb < c_0 < 0$.

<u>Claim</u>: $c^Tz \geq 0$ for all $C$.

If not, $c^Tz < 0 \Rightarrow \lambda c^Tz = c^T(\lambda z) < c_0$ for some $\lambda \geq 0$.

But $C$ is a convex cone $\Rightarrow \lambda z \in C \Rightarrow c^T(\lambda z) > c_0 \ \forall \lambda \geq 0$.

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\rightarrow$ take $y = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$




$y^TA = (2, 1, 3) \geq 0$

$y^Tb = -1 < 0$

Many other incarnations of Farkas' Lemma:

ORIGINAL:

1) $\exists x \in \mathbb{R}^n_{\geq 0}$ s.t. $Ax = b \Leftrightarrow \nexists y \in \mathbb{R}^m$ s.t. $y^T A \in \mathbb{R}^n_{\geq 0}$, $y^T b < 0$

polytope $P = \{x \in \mathbb{R}^n_{\geq 0} : Ax = b\}$ nonempty $\Leftrightarrow$ $y^T b \geq 0$ on $\{y \in \mathbb{R}^m : y^T A \geq 0\}$

2) $\exists x \in \mathbb{R}^n$ s.t. $Ax \leq b \Leftrightarrow \nexists y \in \mathbb{R}^m_{\geq 0}$ s.t. $y^T A = 0$, $y^T b < 0$

polytope $P = \{x : Ax \leq b\}$ nonempty $\Leftrightarrow$ $y^T b \geq 0$ on $\{y \in \mathbb{R}^m_{\geq 0} : y^T A = 0\}$

3) $\exists x \in \mathbb{R}^n_{\geq 0}$ s.t. $Ax \leq b \Leftrightarrow \nexists y \in \mathbb{R}^m_{\geq 0}$ s.t. $y^T A \geq 0$, $b^T y < 0$

polytope $P = \{x \in \mathbb{R}^n_{\geq 0} : Ax \leq b\}$ nonempty $\Leftrightarrow$ $y^T b \geq 0$ on $\{y \in \mathbb{R}_{\geq 0} : y^T A \geq 0\}$

(Proof of 2)

For $x \in \mathbb{R}^n$, $Ax \leq b \Leftrightarrow \exists s \in \mathbb{R}^m_{\geq 0}$ s.t. $Ax + s = b$.

Write $x = x^+ - x^-$ with $x^+, x^- \in \mathbb{R}^n_{\geq 0}$, $Ax = Ax^+ - Ax^-$

All together, this gives that

$\exists x \in \mathbb{R}^n$ s.t. $Ax \leq b$

$\Leftrightarrow \exists x^+, x^- \in \mathbb{R}^n_{\geq 0}, s \in \mathbb{R}^m_{\geq 0}$ s.t. $Ax^+ - Ax^- + s = b$

$\Leftrightarrow \exists \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} \in \mathbb{R}^{2n+m}_{\geq 0}$ s.t. $\begin{bmatrix} A & -A & I_m \end{bmatrix} \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} = b$

$\overset{(1)}{\Leftrightarrow} \nexists y \in \mathbb{R}^m$ s.t. $y^T \begin{bmatrix} A & -A & I_m \end{bmatrix} \geq 0$, $y^T b < 0$

$\Leftrightarrow \nexists y \in \mathbb{R}^m$ s.t. $\underline{y^T A \geq 0 \;\; y^T(-A) \geq 0}$, $y \geq 0$, $y^T b < 0$

$\Leftrightarrow \nexists y \in \mathbb{R}^m$ s.t. $\underline{y^T A = 0}$, $y \geq 0$, $y^T b < 0$.

(See notes/hwk problems for more)

<u>Cor 2.5b.</u> Suppose $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is nonempty.
Then $c^T x \leq c_0$ for all $x \in P$ $\iff$ $\exists\, y \in \mathbb{R}^m_{\geq 0}$ s.t. $y^T A = c^T$, $y^T b \leq c_0$

★ Every inequality valid on P is a conic comb. of $a_1^T x \leq b_1, \dots, a_m^T x \leq b_m$ ★

($\Leftarrow$) $Ax \leq b$, $y \in \mathbb{R}^m_{\geq 0}$ $\Rightarrow$ $c^T x = y^T A x \leq y^T b \leq c_0$

($\Rightarrow$) Suppose $\nexists$ such y. That is $\nexists \begin{pmatrix} y \\ y_0 \end{pmatrix} \in \mathbb{R}^{m+1}_{\geq 0}$ s.t.

$$(y^T \;\; y_0)\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} = (c^T \;\; c_0)$$

By Farkas' Lemma, $\exists \begin{pmatrix} z \\ z_0 \end{pmatrix} \in \mathbb{R}^{n+1}$ s.t. ← (Transpose and variable switch from usual version)

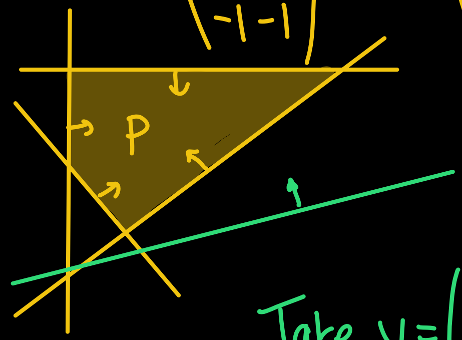
$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z \\ z_0 \end{pmatrix} = \begin{pmatrix} Az + bz_0 \\ z_0 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } (c^T \;\; c_0)\begin{pmatrix} z \\ z_0 \end{pmatrix} = c^T z + c_0 z_0 < 0.$$

<u>Case 1</u> ($z_0 = 0$) Then $Az \geq 0$ and $c^T z < 0$. Let $x \in P$.
For all $\lambda \geq 0$, $A(x - \lambda z) = Ax - \lambda Az \leq Ax \leq b \Rightarrow x - \lambda z \in P$.
but for large enough $\lambda$, $c^T(x - \lambda z) = c^T x - \lambda c^T z < 0$
$\Rightarrow c^T x \leq c_0$ does not hold for all $x \in P$.

<u>Case 2</u> ($z_0 > 0$) We can rescale so that $z_0 = 1$ $\left(\begin{pmatrix} z \\ z_0 \end{pmatrix} \to \begin{pmatrix} (1/z_0) z \\ 1 \end{pmatrix}\right)$
Then $Az + b \geq 0 \Rightarrow A(-z) \leq b \Rightarrow -z \in P$.
Also $c^T z + c_0 < 0 \Rightarrow -c^T z - c_0 > 0 \Rightarrow c^T(-z) > c_0$
$\Rightarrow c^T x \leq c_0$ does not hold for all $x \in P$ (e.g. $x = -z$)

Ex: $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & -1 \\ -1 & -1 \end{pmatrix}$ $b = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ $P = \{x \in \mathbb{R}^2 : Ax \le b\}$

$= \{x \in \mathbb{R}^2 : 0 \le x_1,\ x_2 \le 1,\ x_1 - x_2 \le 1,\ x_1 + x_2 \ge 0\}$



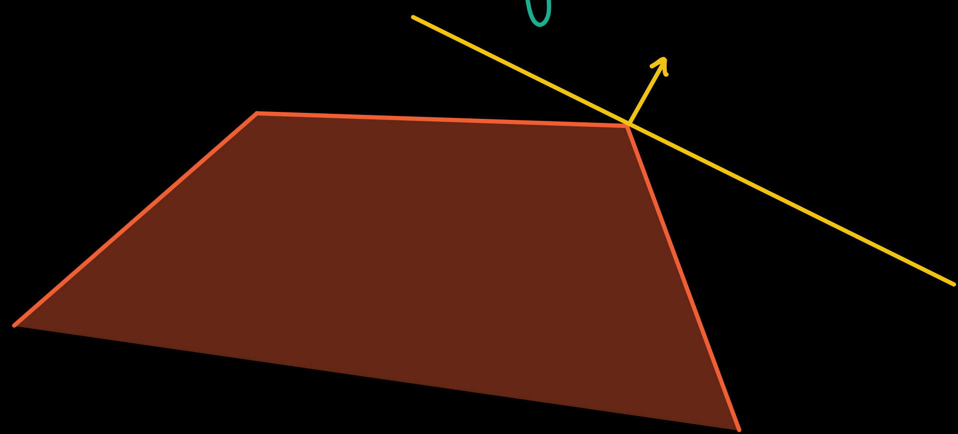

$x_1 - 2x_2 \le 2$ for all $x \in P$

$\Rightarrow \exists\, y \in \mathbb{R}^4_{\ge 0}$ s.t. $y^T A = (1\ -2)$ $y^T b \le -1$

Take $y = \begin{pmatrix} 0 \\ 0 \\ 3/2 \\ 1/2 \end{pmatrix}$ $y^T A = \frac{3}{2}(1,-1) + \frac{1}{2}(-1,-1) = (1\ -2)$

$y^T b = \frac{3}{2}(1) + \frac{1}{2}(0) = \frac{3}{2} \le 2$ ✓

# Geometric Insights




The point $y^* \in \mathbb{R}^m$ attaining $\min\{y^T b : y \ge 0, y^T A = c^T\} = \delta$ writes $c^T x \le \delta$ as a nonneg. linear combination of the inequalities $a_1^T x \le b_1, \ldots, a_m^T x \le b_m$

$y^* \ge 0,\ (y^*)^T A = c^T,\ (y^*)^T b = \delta$

$$\Rightarrow \begin{matrix} y_1^* a_1^T x \le y_1^* b_1 \\ \vdots \\ y_m^* a_m^T x \le y_m^* b_m \end{matrix} \Rightarrow \underbrace{\sum_{i=1}^m y_i^* a_i^T x}_{= c^T x} \le \underbrace{\sum_{i=1}^m y_i^* b_i}_{= \delta}$$