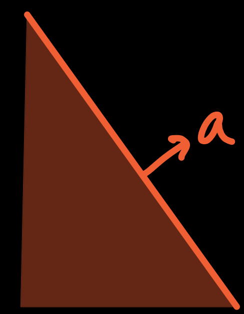
MA/AMA 514:

Networks & Combinatorial Optimization

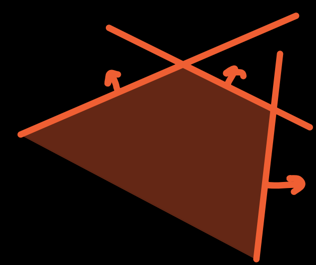
Today: §2.2, §2.4

§2.2 Polytopes and Polyhedra

A closed halfspace is a set of the form $\{x \in \mathbb{R}^n : a^T x \leq b\}$ where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$



A polyhedron is the intersection of finitely-many halfspaces, i.e.



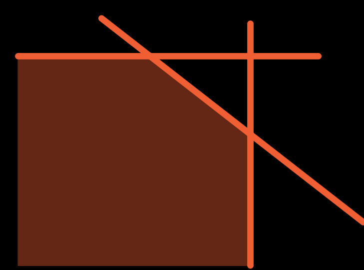
$$P = \{x \in \mathbb{R}^n : a_1^T x \leq b_1, \ldots, a_m^T x \leq b_m\}$$

where $a_1, \ldots, a_m \in \mathbb{R}^n$, $b_1, \ldots, b_m \in \mathbb{R}$

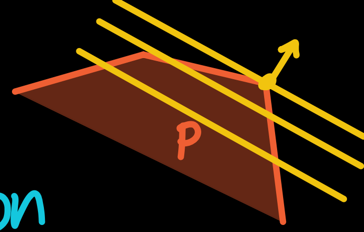
We write this as $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ where

$$A = \begin{bmatrix} - & a_1 & - \\ & \vdots & \\ - & a_m & - \end{bmatrix} \in \mathbb{R}^{m \times n} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$$

Ex: $Ax = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = b$



Linear programming



The problem of maximizing a linear function over a polyhedron is known as a <u>linear program</u> (LP)

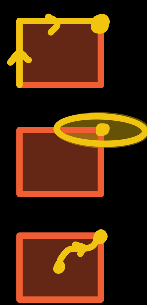
One standard form: $\max\{c^T x : Ax \le b\}$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

Polyhedron $P = \{x : Ax \le b\} \subseteq \mathbb{R}^n$, defined by m inequalities

There are many methods for solving LP's:

- simplex method (Dantzig, 1951)
- ellipsoid method (Khachiyan, 1979)
- interior point methods (Karmarkar, 1984)

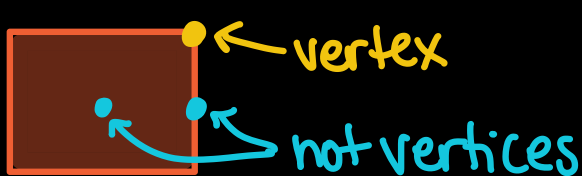


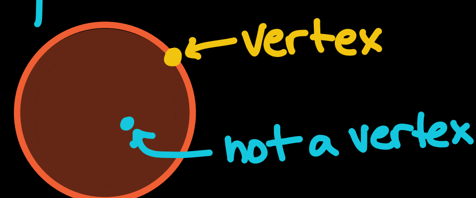
first polynomial time alg. for solving LP's

What points attain the optimal value?

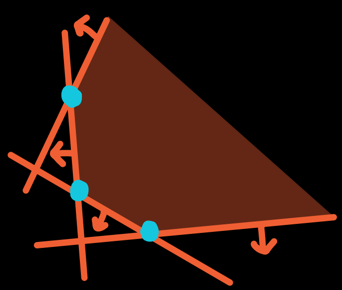
Let C be convex. Call $z \in C$ a <u>vertex</u> of C if there do not exist $x, y \in C \setminus \{z\}$ and $\lambda \in (0,1)$ s.t.

$$z = \lambda x + (1-\lambda) y$$





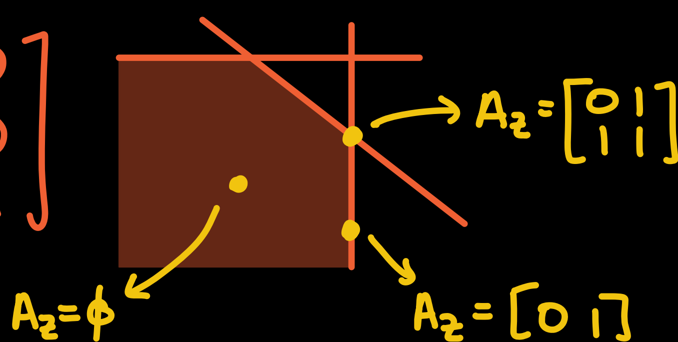
Vertices of Polyhedra



In $\mathbb{R}^2$, it seems that we need two equalities $a_i^T z = b_i$, $a_j^T z = b_j$ to get a vertex

More generally, for a polyhedron $P = \{x \in \mathbb{R}^n : Ax \le b\}$ and a point $z \in P$, let A_z be the submatrix of A with rows $\{a_i : a_i^T z = b\}$

Ex: $Ax = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

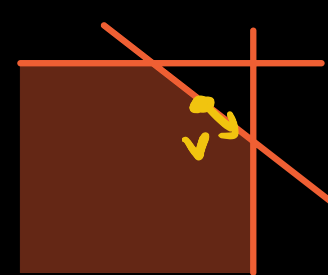


Thm 2.2 Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$ be a polyhedron. A point $z \in P$ is a vertex of P $\iff$ $\text{rank}(A_z) = n$

(Idea of $\Rightarrow$) $\text{rank}(A_z) < n$

$\Rightarrow$ $\exists v \in \mathbb{R}^n \setminus \{0\}$ s.t. $A_z v = 0$.

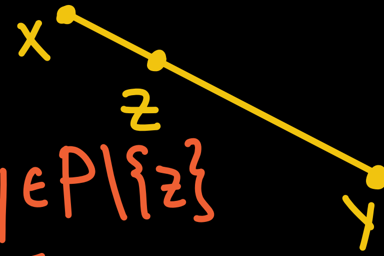
$\Rightarrow$ for small enough $\varepsilon > 0$, $z \pm \varepsilon v \in P$



Then $z = \frac{1}{2}(\underbrace{z + \varepsilon v}) + \frac{1}{2}(\underbrace{z - \varepsilon v}) \Rightarrow z$ not a vertex

both in P

(Idea of $\Leftarrow$) z not a vertex



$\Rightarrow z = \lambda x + (1-\lambda) y$ for some $\lambda \in (0,1)$, $x, y \in P \setminus \{z\}$

$a_i^T x \leq b_i$ and $a_i^T y \leq b_i$ for all $i \in [m]$,

$\Rightarrow a_i^T z = \lambda a_i^T x + (1-\lambda) a_i^T y \leq \lambda b_i + (1-\lambda) b_i = b_i$

with equality $\Leftrightarrow a_i^T x = b_i$ and $a_i^T y = b_i$

$\Rightarrow a_i^T (x-y) = 0$ whenever $a_i^T z = b_i$.

$\Rightarrow A_z (x-y) = 0 \Rightarrow \operatorname{rank}(A_z) < n$.

Cor: Every polyhedron has finitely-many vertices.

(Proof) A vertex z of P is the unique solution to $A_z x = b_z$ (where $b_z = (b_i)_{i \in I}$).

If $z \neq w$ are vertices of P, then $A_z \neq A_w$.

That is, $\{i \in [m] : a_i^T z = b_i\} \neq \{i \in [m] : a_i^T w = b_i\}$.

Only 2^m distinct subsets of $[m] \Rightarrow \leq 2^m$ vertices of P.

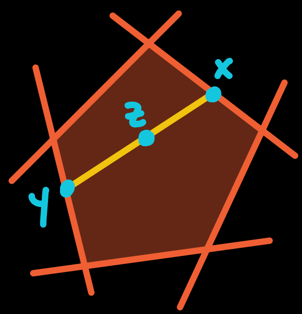
Polytopes

A <u>polytope</u> is the convex hull of a finite set. $\square$

<u>Thm 2.3</u> A bounded polyhedron is the convex hull of its vertices.

(See Krein-Milman Thm for more general convex sets)

(Idea of proof) Let P be a bounded polyhedron with vertices $x_1, \dots, x_t$.



Let $z \in P$. WTS $z \in \operatorname{conv}(\{x_1, \dots, x_m\})$.

Induct on $n - \operatorname{rank}(A_z)$.

If $\operatorname{rank}(A_z) = n$, z is a vertex of P. ✓

If $\operatorname{rank}(A_z) = n$, $\exists v \in \mathbb{R}^n \setminus \{0\}$ s.t. $A_z v = 0$.

P bounded $\Rightarrow \{z + \lambda v : \lambda \in \mathbb{R}\} \cap P$ is a bounded line segment. Let x, y be the end points.

Then $z = \lambda x + (1-\lambda) y$ for some $\lambda \in (0,1)$.

Moreover, x, y are endpoints because of some new ineq. holding with equality at x or y.

$\Rightarrow \operatorname{rank}(A_x), \operatorname{rank}(A_y) > \operatorname{rank}(A_z)$

$\underset{\text{(Induction)}}{\Longrightarrow}$ $x, y \in \operatorname{conv}\{x_1, \dots, x_m\} \Rightarrow z \in \operatorname{conv}\{x_1, \dots, x_m\}$.

Converse also holds! (See Thm 2.4)

Cor 2.3a. A bounded polyhedron is a polytope.

$P \subseteq \mathbb{R}^n$ is a bounded polyhedron $\Leftrightarrow$ P is a polytope

Claim: If $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is bounded then $\max\{c^T x : x \in P\}$ is attained by some vertex of P.

(Proof) Since P is compact, $\max\{c^Tx : x\in P\}$ is attained by some $z\in P$. By Thm 2.3, $z\in \operatorname{conv}\{x_1,\dots,x_m\}$.

$\Rightarrow z=\sum_{i=1}^m \lambda_i x_i$ where $\lambda_1,\dots,\lambda_m\geq 0$, $\sum \lambda_i = 1$.

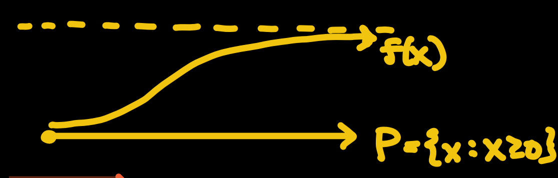
If $c^Tx_i<\mu$ for all i, $c^Tz=\sum_i \lambda_i c^Tx_i < \sum_i \lambda_i\mu=\mu$

$\Rightarrow$ If $c^Tz=\mu=\max\{c^Tx : x\in P\}$ then $c^Tx_i=\mu$ for some i.

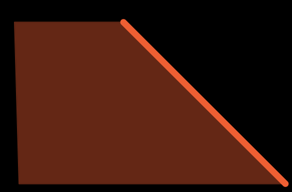
A stronger statement holds:

<u>Lemma 2.1</u> If $\sup\{c^Tx : Ax\leq b\}<\infty$, then $\max\{c^Tx : Ax\leq b\}$ is attained.

Note: this isn't always true for nonlinear objective functions!



Ex: max $x+y$ s.t. $x+y\leq 1$



max is attained, not at a vertex