MA/AMA 514:
Networks & Combinatorial Optimization

Today: §1.4, §2.1, §2.2

From last time:

Given a connected graph $G=(V,E)$ and length function $\ell: E \to \mathbb{R}$, want to find minimum length spanning tree:

$$\ell(T) = \sum_{e \in T} \ell(e).$$

Thm 1.11 Suppose

- F is a greedy forest of G
- U is the vertex set of a conn. component of F
- $e \in \delta(U)$ has min. length in $\delta(U)$

then $F \cup \{e\}$ is a greedy forest of G.

Dijkstra-Prim Algorithm

(1959, 1957, based on Borůvka (1926))

1) Choose any vertex $v \in V$. Set $V_0 = \{v\}$, $E_0 = \emptyset$

2) While $V_k \neq V$, choose $e_k \in \delta(V_k)$ of min. length
Set $V_{k+1} = V_k \cup e_k$ and $E_{k+1} = E_k \cup \{e_k\}$.

(Proof) Start: $E_0 = \phi$ is a greedy forest
By Thm, every update E_k is also.

$\Rightarrow$ Output = a spanning, connected greedy forest
= a min. spanning tree

Kruskal's Algorithm (1956)

1) Sort edges s.t. $\ell(e_1) \leq \ell(e_2) \leq \ldots \leq \ell(e_m)$ $(|E|=m)$

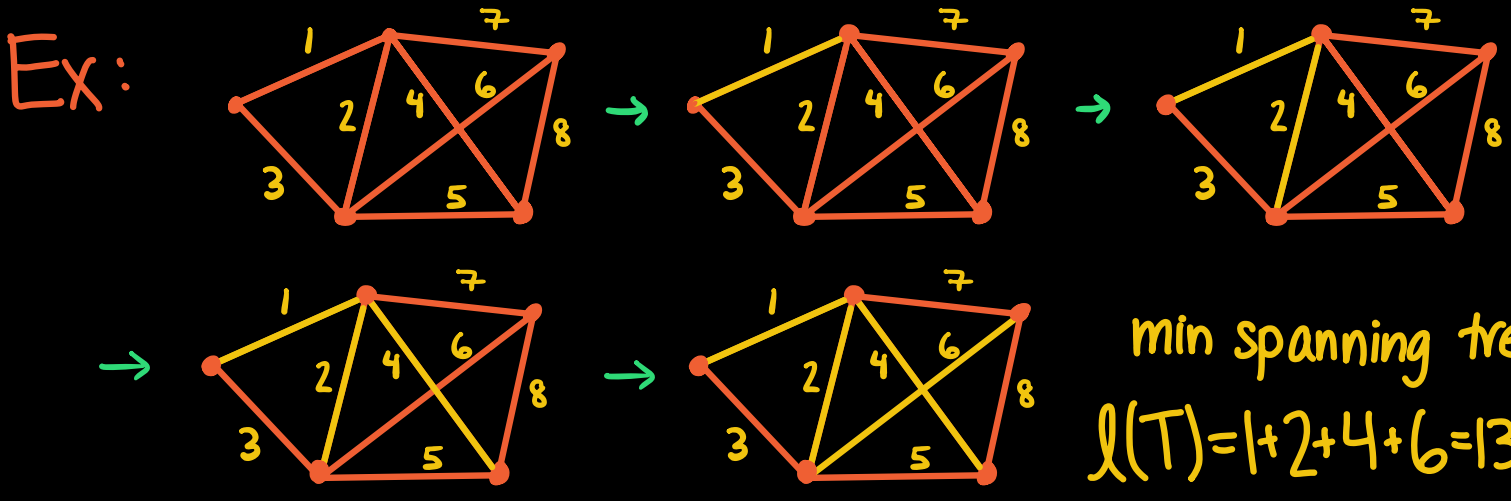
2) Set $F_0 = (V, \phi)$

3) For $i = 1, \ldots, m$, if $F_{i-1} \cup \{e_i\}$ acyclic, $F_i = F_{i-1} \cup \{e_i\}$
otherwise $F_i = F_{i-1}$.

Output $T = F_m$.

Run time:
$O(|E| \log(|E|))$
sorting edges!

Ex:



min spanning tree
$\ell(T) = 1+2+4+6 = 13$

(Proof of correctness) Start F_0 is a greedy forest

If $F_{i-1} \cup \{e_i\}$ is acyclic, e_i connects two connected components U, U' of $F_{i-1} \Rightarrow e_i \in \delta(U)$.

If $\ell(f) < \ell(e_i)$, end pts of f are connected in F_{i-1}.
$\Rightarrow e_i$ has min length among $\delta(U)$.
$\Rightarrow$ By Thm 1.1, F_i is a greedy forest.

Application 1.7: Maximum Reliability Problem

Want to design a network that maximizes "reliability"

Input: Graph $G=(V,E)$, $s: E \to \mathbb{R}_+$

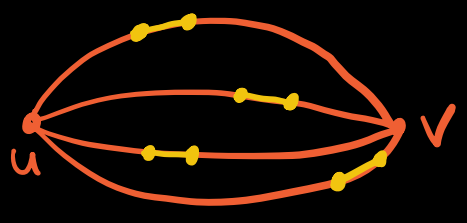
$s(e)$ = "strength" of edge

The reliability of a path P is $r(P) = \min_{e \in E(P)} s(e)$

(strength of weakest link in P) →

For vertices $u, v \in V$, define

$r_G(u,v) = \max\{r(P) : P \text{ is a path in } G \text{ from } u \text{ to } v\}$

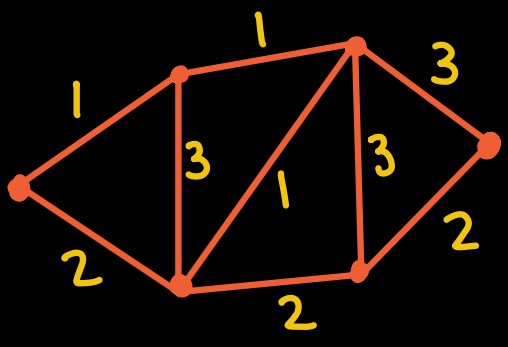


(Strongest weakest link!)

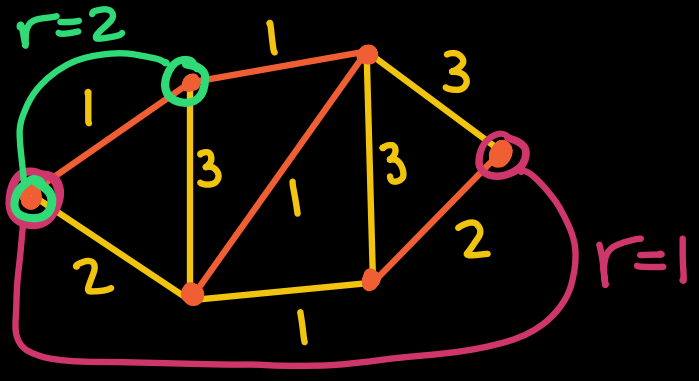
Let T be a maximum strength spanning tree (minimum length with $\ell(e) = -s(e)$).

Claim: $r_G(u,v) = r_T(u,v)$ for all $u,v \in V$

Proof in Hwk 1!



Kruskal →



Greedy Algorithms & Matroids

"Greedy algorithms" don't always find the optimal solution! Formally..

Problem: Given a finite set X, collection $\mathcal{B}$ of subsets of X, and weights $w: X \to \mathbb{R}$, find the minimum weight element of $\mathcal{B}$, where

$$w(S) = \sum_{x \in S} w(x).$$

Ex: X = edges of a connected graph G

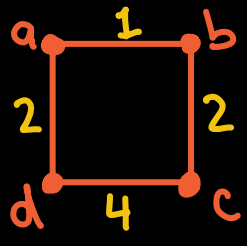
$\mathcal{B}$ = {spanning trees of G}

Greedy Algorithm Start: $S = \emptyset$

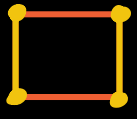
While $S \notin \mathcal{B}$, choose $x \in X \setminus S$ of minimum weight so that $S \cup \{x\} \subseteq B$ for some $B \in \mathcal{B}$ and update $S \to S \cup \{x\}$.

If there is no such x, stop and output S.

Ex (failure!) Find the minimum length spanning set of $G = (V, E)$, $\ell : E \to \mathbb{R}$



Greedy alg: Choose shortest edge (a,b) try to add to it...

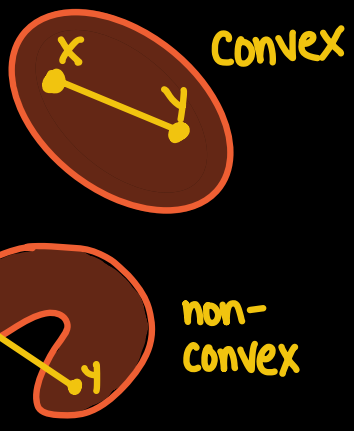
Only optimal solution  doesn't use ab

Structures for which the greedy alg. always finds the optimal solution are called matroids.

(discussed in §10, later in the quarter!)

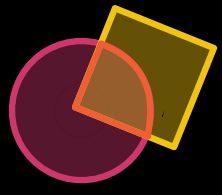
§2.1/§2.2 Convex Sets

A set $C \subseteq \mathbb{R}^n$ is <u>convex</u> if
for every $x, y \in C$, $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in C$
That is, C contains the line segment between any two points in C.



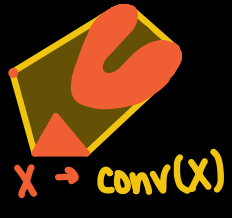
Remark: The intersection of convex sets is convex.
$C_i \subseteq \mathbb{R}^n$ convex for all $i \in I \Rightarrow \bigcap_{i \in I} C_i$ convex

$x, y \in \bigcap_i C_i \Rightarrow x, y \in C_i \ \forall i \in I$
$\Rightarrow \lambda x + (1-\lambda)y \in C_i \ \forall i \in I$
$\Rightarrow \lambda x + (1-\lambda)y \in \bigcap_{i \in I} C_i$



The <u>convex hull</u> of a set $X \subseteq \mathbb{R}^n$, denoted $\text{conv}(X)$ is the smallest convex set containg X, i.e.

$$\text{conv}(X) = \bigcap_{\substack{C \text{ convex} \\ X \subseteq C}} C$$



Alternatively,

$$\text{conv}(X) = \left\{ \sum_{i=1}^m \lambda_i x_i : x_1, \ldots, x_m \in X,\ \lambda_i \geq 0,\ \sum_{i=1}^m \lambda_i = 1 \right\}$$

↖ a "convex combination" of $x_1, \ldots, x_m$

The convex hull of a finite set is a <u>polytope</u>

Ex 1: $X = \{(0,0), (1,0), (0,1)\}$

$\Rightarrow \text{Conv}(X) = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ 1 \geq x+y\}$

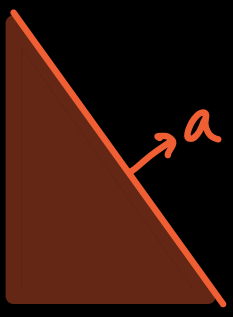
$= (1-x-y)(0,0) + x(1,0) + y(0,1)$

coeff are ≥ 0 and sum to 1

Ex 2: $X = \{(0,0,0)\} \cup \{(x,y,1) : x^2+y^2 = 1\}$

$\text{Conv}(X) = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq z^2,\ 0 \leq z \leq 1\}$

Ex/Def: a closed halfspace is a set of the form $\{x \in \mathbb{R}^n : a^T x \leq b\}$ where $a \in \mathbb{R}^n$, $b \in \mathbb{R}$

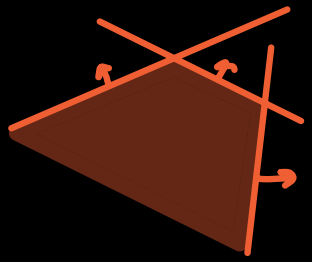


Convexity: $a^T x \leq b$, $a^T y \leq b \Rightarrow \lambda a^T x \leq \lambda b$

$(1-\lambda) a^T y \leq (1-\lambda) b$

$\Rightarrow a^T(\lambda x + (1-\lambda) y) \leq b$

A <u>polyhedron</u> is the intersection of finitely-many halfspaces, i.e.



$P = \{x \in \mathbb{R}^n : a_1^T x \leq b_1, \ldots, a_m^T x \leq b_m\}$

where $a_1, \ldots, a_m \in \mathbb{R}^n$, $b_1, \ldots, b_m \in \mathbb{R}$

Ex: $\{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ 1 \geq x+y\}$

Thm: Every bounded polyhedron is a polytope. (Cor. 2.3a)

Thm 2.4 Every polytope is a bounded polyhedron.

Teaser for the future:

Spanning trees as a polytope

$G=(V,E)$ connected graph, $\ell: E \to \mathbb{R}$

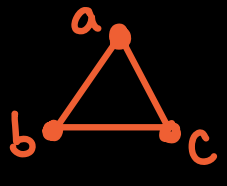
For $T \subseteq E$, define $\mathbb{1}_T \subseteq \mathbb{R}^E$ by

$$(\mathbb{1}_T)_e = \begin{cases} 1 & \text{if } e \in T \\ 0 & \text{if } e \notin T \end{cases}$$

Then $\text{conv}\{\mathbb{1}_T : T \text{ is a span. tree of } G\} = P_G$

is a polytope.

Ex:



$T=\{ab, ac\} \to \mathbb{1}_T = (1,1,0)$ (ab ac bc)

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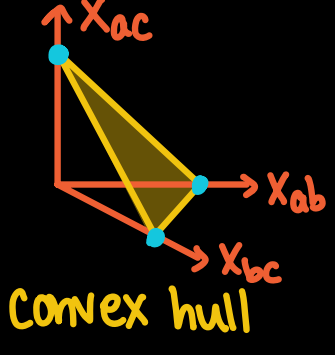

Claim: length of min sp. tree

$$= \min\left\{ \sum_{e \in E} \ell(e) x_e : x \in P_G \right\}$$