MA/AMA 514:
Networks & Combinatorial Optimization

Today: §1.4, §2.1, §2.2

From last time:

Given a connected graph $G=(V,E)$ and length function $\ell: E \to \mathbb{R}$, want to find minimum length spanning tree:

$$\ell(T) = \sum_{e \in T} \ell(e).$$

<u>Thm 1.11</u> Suppose

- F is a greedy forest of G
- U is the vertex set of a conn. component of F
- $e \in \delta(U)$ has min. length in $\delta(U)$

then $F \cup \{e\}$ is a greedy forest of G.

<u>Kruskal's Algorithm (1956)</u>

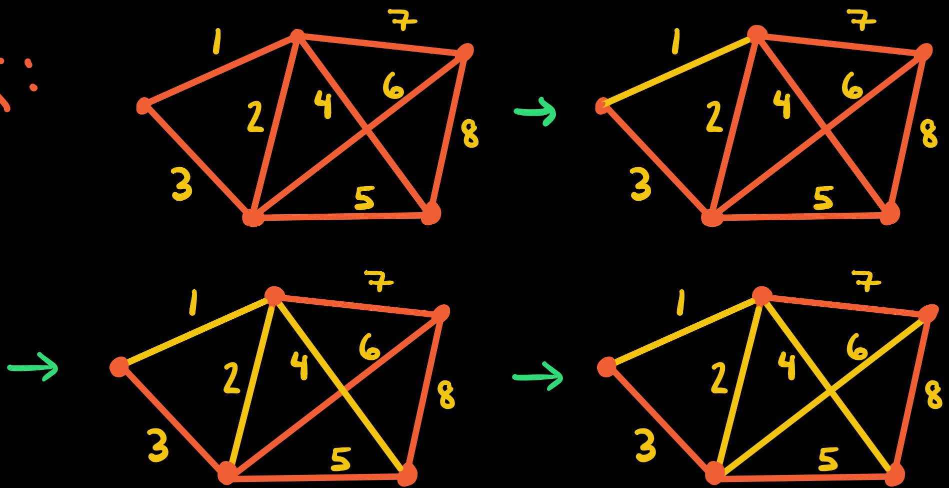
1) Sort edges s.t. $\ell(e_1) \leq \ell(e_2) \leq \dots \leq \ell(e_m)$ $\quad (|E| = m)$
2) Set $F_0 = (V, \phi)$

3) For $i=1,\dots,m$, if $F_{i-1}\cup\{e_i\}$ acyclic, $F_i=F_{i-1}\cup\{e_i\}$
otherwise $F_i=F_{i-1}$.

Output $T=F_m$.

Run time: $O(|E|\log(|E|))$
sorting edges!

Ex:

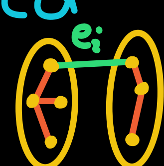


min spanning tree
$\ell(T)=1+2+4+6=13$

(Proof of correctness) Start F_0 is a greedy forest.
If $F_{i-1}\cup\{e_i\}$ is acyclic, e_i connects two connected components $\mathcal{U},\mathcal{U}'$ of F_{i-1} $\Rightarrow e_i\in\delta(\mathcal{U})$.

If $\ell(f)<\ell(e_i)$, end pts of f are connected in F_{i-1}.
$\Rightarrow e_i$ has min length among $\delta(\mathcal{U})$.
$\Rightarrow$ By Thm 1.1, F_i is a greedy forest.

Q: How do we implement step (3)?
One way is to keep track of the connected components of F_i. $F_{i-1}\cup\{e_i\}$ acyclic $\Leftrightarrow$ endpts of e_i lie in different conn. comp.



Application 1.7: Maximum Reliability Problem

Want to design a network that maximizes "reliability"

Input: Graph $G=(V,E)$, $s: E \to \mathbb{R}_+$

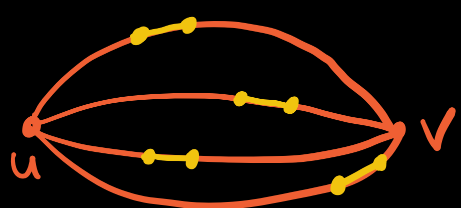
$s(e)$ = "strength" of edge

The reliability of a path P is $r(P) = \min_{e \in E(P)} s(e)$

(strength of weakest link in P) →

For vertices $u,v \in V$, define

$r_G(u,v) = \max\{r(P) : P \text{ is a path in } G \text{ from } u \text{ to } v\}$.

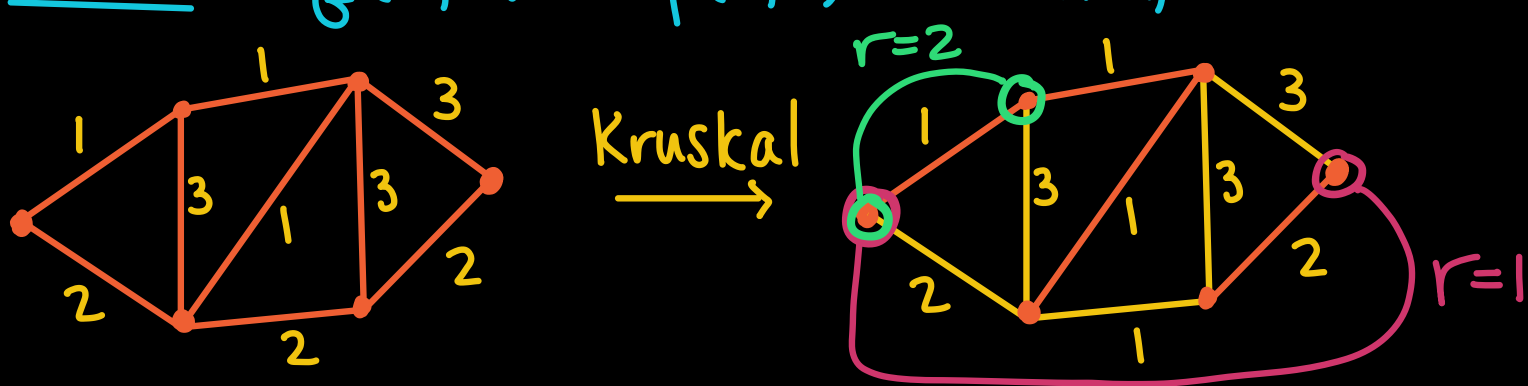


(strongest weakest link!)

Let T be a maximum strength spanning tree (minimum length with $\ell(e) = -s(e)$).

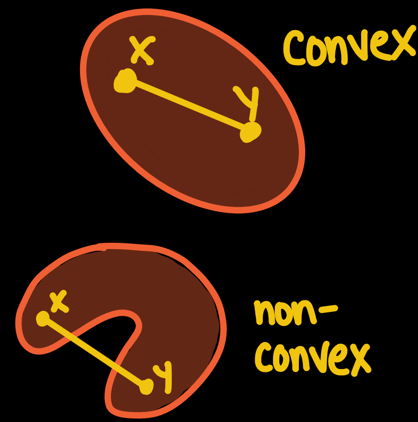
Claim: $r_G(u,v) = r_T(u,v)$ for all $u,v \in V$

Proof in Hwk 1!

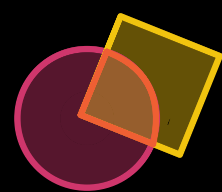


§2.1/§2.2 Convex Sets

A set $C \subseteq \mathbb{R}^n$ is <u>convex</u> if
for every $x, y \in C$, $\lambda \in [0,1]$, $\lambda x + (1-\lambda) y \in C$
That is, C contains the line segment between any two points in C.



Remark: The intersection of convex sets is convex.
$C_i \subseteq \mathbb{R}^n$ convex for all $i \in I \Rightarrow \bigcap_{i \in I} C_i$ convex



$x, y \in \bigcap_i C_i \Rightarrow x, y \in C_i \ \forall i \in I$
$\Rightarrow \lambda x + (1-\lambda) y \in C_i \ \forall i \in I$
$\Rightarrow \lambda x + (1-\lambda) y \in \bigcap_{i \in I} C_i$

The <u>convex hull</u> of a set $X \subseteq \mathbb{R}^n$, denoted $\mathrm{conv}(X)$ is the smallest convex set containg X, i.e.

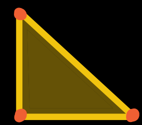
$$\mathrm{conv}(X) = \bigcap_{\substack{C \text{ convex} \\ X \subseteq C}} C$$

Alternatively,

$$\mathrm{conv}(X) = \left\{ \sum_{i=1}^{m} \lambda_i x_i : x_1, \ldots, x_m \in X, \ \lambda_i \geq 0, \ \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

← a "convex combination" of $x_1, \ldots, x_m$

The convex hull of a finite set is a <u>polytope</u>

Ex 1: $X=\{(0,0),(1,0),(0,1)\}$ 

$\Rightarrow \text{Conv}(X)=\{(x,y)\in\mathbb{R}^2 : x\geq 0,\ y\geq 0,\ 1\geq x+y\}$

$\subseteq = \underbrace{(1-x-y)}(0,0)+\underbrace{x}(1,0)+\underbrace{y}(0,1)$

coeff are ≥ 0 and sum to 1

Ex 2: $X=\{(0,0,0)\}\cup\{(x,y,1): x^2+y^2=1\}$ 

$\text{Conv}(X)=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\leq z^2,\ 0\leq z\leq 1\}$

Claim: For $X\subseteq\mathbb{R}^n$, $c\in\mathbb{R}^n$,

$$\max\{c^Tx : x\in X\} = \max\{c^Ty : y\in \text{conv}(X)\}$$

(Proof) "$\leq$" follows from $X\subseteq\text{conv}(X)$.

For "$\geq$", let $c_0=\max\{c^Tx : x\in X\}$ and $y\in\text{conv}(X)$

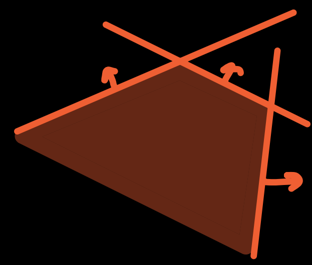
$\Rightarrow y=\sum_{i=1}^m \lambda_i x_i$ for some $x_1,\dots,x_m\in X$, $\lambda_i\geq 0$ s.t. $\sum\lambda_i=1$

$$\Rightarrow c^Ty \underset{\text{linearity}}{=} \sum_{i=1}^m \lambda_i c^Tx_i \underset{\lambda_i\geq 0\ \forall i}{\leq} \sum_{i=1}^m \lambda_i c_0 \underset{\sum\lambda_i=1}{=} c_0.$$

Ex/Def: a closed halfspace is a set of the form $\{x\in\mathbb{R}^n : a^Tx\leq b\}$ where $a\in\mathbb{R}^n$, $b\in\mathbb{R}$

Convexity: $a^Tx\leq b,\ a^Ty\leq b \Rightarrow$
$$\begin{array}{r} \lambda a^Tx\leq\lambda b \\ (1-\lambda)a^Ty\leq(1-\lambda)b \\ \hline \end{array}$$
$$\Rightarrow a^T(\lambda x+(1-\lambda)y)\leq b$$

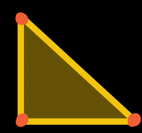
A polyhedron is the intersection of finitely-many halfspaces, i.e.



$$P = \{x \in \mathbb{R}^n : a_1^T x \le b_1, \ldots, a_m^T x \le b_m\}$$

where $a_1, \ldots, a_m \in \mathbb{R}^n$, $b_1, \ldots, b_m \in \mathbb{R}$

Ex: $\{(x,y) \in \mathbb{R}^2 : x \ge 0,\ y \ge 0,\ 1 \ge x+y\}$



Thm 2.4 Every polytope is a bounded polyhedron.

A set $X \subseteq \mathbb{R}^n$ is bounded if X is contained in some ball, i.e. $\exists R \in \mathbb{R}_{>0}$ s.t. $\|x\|_2 \le R$ for all $x \in X$.

You check: If X is bounded, so is $\mathrm{conv}(X)$.

Thm: Every bounded polyhedron is a polytope. (Cor. 2.3a)

We will translate comb. opt. problems to the form $\max\{c^T x : x \in X\}$ with X finite and then replace X with $\mathrm{conv}(X)$.

Teaser for the future:

Spanning trees as a polytope

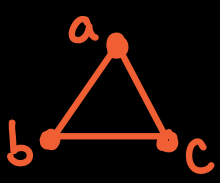
$G = (V, E)$ connected graph, $\ell : E \to \mathbb{R}$

For $T \subseteq E$, define $\mathbb{1}_T \in \mathbb{R}^E$ by

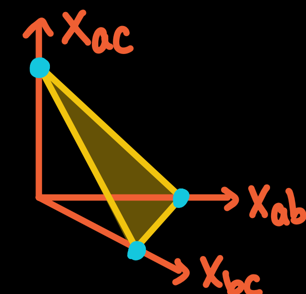
$$(\mathbb{1}_T)_e = \begin{cases} 1 & \text{if } e \in T \\ 0 & \text{if } e \notin T \end{cases}$$

Then $\text{conv}\{\mathbb{1}_T : T \text{ is a span. tree of } G\} = P_G$ is a polytope.

Ex:



$$\begin{array}{llll} & & & \quad\ \ ab\ \ ac\ \ bc \\ T = \{ab, ac\} & \rightarrow & \mathbb{1}_T = & (1,1,0) \\ T = \{ab, bc\} & \rightarrow & \mathbb{1}_T = & (1,0,1) \\ T = \{ac, bc\} & \rightarrow & \mathbb{1}_T = & (0,1,1) \end{array}$$



convex hull

Claim: length of min sp. tree

$$= \min\left\{ \sum_{e \in E} l(e) x_e : x \in P_G \right\}$$