Math 409 – Homework 3

Due on Thursday, April 18

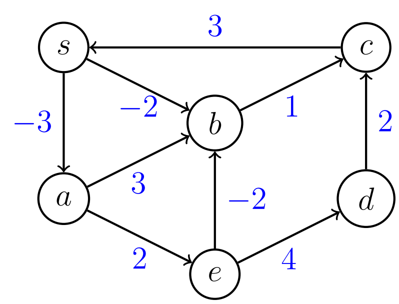
You are welcome to talk with other students in the class about problems but should write up solutions on your own. Solutions can be handwritten or typed but need to be legible and submitted via Gradescope by the end of the day on Thursday. You should justify all your answers in order to receive full credit.

Problem 1. Let $G = (V, E)$ be a directed graph with edge costs $c : E \to \mathbb{R}$ and suppose that $P = \{(s, v_1), (v_1, v_2), \ldots, (v_k, t)\}$ is the shortest $s-t$ path in G.

(a) Fill in the details on the proof of Bellman's principal (Lemma 9). That is, show that if G has no negative cost cycles, then for each $i \le k$, $P_i = \{(s, v_1), (v_1, v_2), \ldots, (v_{i-1}, v_i)\}$ is the shortest $s - v_i$ path in G. (*Hint:* it may be useful to show that the minimum cost of an $s-t$ *walk* in G achieved by a path.)

(b) Show that this no longer holds without the assumption of no negative cycles. That is, give an example of a graph $G = (V, E)$, edge costs $c : E \to \mathbb{R}$, and shortest $s-t$ path P so that the $s - v_i$ portion of P is not the shortest $s - v_i$ path in G.

Problem 2. Consider the following directed graph $G = (V, E)$ (with edge costs in blue).



(a) Use the Moore-Bellman-Ford algorithm to compute the distances from s to each other node v (call those values $\ell(v)$. It is enough to give the final outcomes $\ell(v)$.

(b) Consider the function $\pi : V \to \mathbb{R}$ given by $\pi(v) = \ell(v)$ using the values of $\ell(v)$ computed in part (a). Re-label the graph with the reduced costs $c_\pi(e)$ from this function. Is π a feasible potential?

c) Run Dijkstra's algorithm with cost function c_π from part (b) and source node a. Use the symbol $\ell'(v)$ to denote the computed a-v distances. It is enough to give the final values of $\ell'(v)$.

d) How do you translate the values $\ell'(v)$ from part (c) into the minimum costs of $a - v$ paths w.r.t the original cost function c? (*Hint:* for a given $a - v$ path P, find a relation between its costs w.r.t. c and c_π.)

Remark: Here you use the more-time-consuming Moore-Bellman-Ford algorithm to compute shortest paths from one source and then use this to be able to use the less-time-consuming Dijkstra's algorithm to compute shortest paths in G from another source (even though the original edge costs are not nonnegative).

Problem 3 on the next page.

Problem 3 (Schrijver)**.** In order to complete a big project, n activities need to be completed, labeled $a_1, \ldots, a_n$. Each activity a_i takes a certain amount of time t_i. Activities can be worked on simultaneously, but some activities need to be completed before others are started. We would like to know the minimum amount of time it will take to complete the project.

Consider the directed graph with $n + 2$ vertices $V = \{s, e, a_1, \ldots, a_n\}$ and directed edges

$$\begin{aligned} E &= \{(s, a_i) : i = 1, \ldots, n\} \cup \{(a_i, e) : i = 1, \ldots, n\} \\ &\quad \cup \{(a_i, a_j) : \text{ activity } a_i \text{ needs to be complete before activity } a_j \text{ starts}\}. \end{aligned}$$

Here s and e are dummy variables representing the start and end of the project, respectively.

(a) Show that the minimum amount to time it will take to complete the project is the *maximum* length cost of an s-e path in the graph (V, E) where the edges (s, a_i) have length zero and edges (a_i, e) and (a_i, a_j) has length t_i.

(b) How would you define edge costs $c : E \to \mathbb{R}$ to solve this as *minimum* cost path problem? Is it reasonable to assume that there are no negative (directed) cycles?

(c) The following activities are part of building a house:

activity	days needed	needs to be done before activity #
1. groundwork	2	2
2. foundation	4	3
3. building walls	10	4,6,7
4. exterior plumbing	4	5,9
5. interior plumbing	5	10
6. electricity	7	10
7. roof	6	8
8. finishing outer walls	7	9
9. exterior painting	9	14
10. panelling	8	11,12
11. floors	4	13
12. interior painting	5	13
13. finishing interior	6	
14. finishing exterior	2	

Find the minimum number of days it will take to complete the house, from start to finish, and describe which activities are bottlenecks. An activity is a *bottleneck* if a delay on the completion of that activity will necessarily increase the completion time of the whole project.

If you solve a min-cost path problem, it is enough to give the final $\ell(v)$ values for all vertices. You do not have to show all your steps in the computation.

Remark: When a directed graph has no directed cycles, one can compute a *topological sorted order* of the vertices, $v_1, \ldots, v_n$ in time $O(|V| + |E|)$ in which $i < j$ whenever $(v_i, v_j) \in E$. Using this, one can modify the Moore-Bellman-Ford algorithm to get an algorithm that finds shortest paths in time $O(|V| + |E|)$ (rather than $O(|V||E|)$).