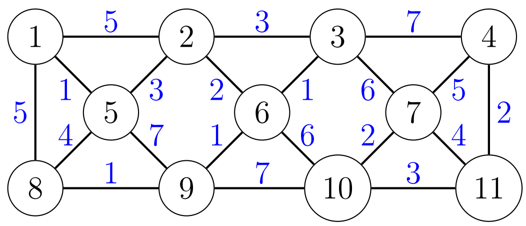
Math 409 – Homework 2

Due on Thursday, April 11

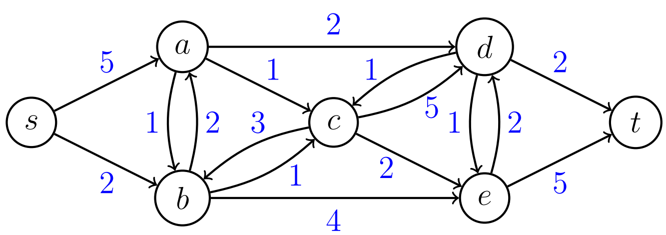
You are welcome to talk with other students in the class about problems but should write up solutions on your own. Solutions can be handwritten or typed but need to be legible and submitted via Gradescope by the end of the day on Thursday. Late submissions can be turned in until the end of Sunday (see the syllabus for late policy). You should justify all your answers in order to receive full credit.

Practice Problem (not graded!). Solve the following discrete optimization problems:

(a) Find a minimum-cost spanning tree of the following graph (edge costs in blue).



(b) Use Dijkstra's Algorithm to compute the shortest path from s to t in the following directed graph (edge lengths in blue).



Problem 1. Let $G = (V, E)$ be an undirected graph with edge costs $c(e) \in \mathbb{R}_{\geq 0}$. Decide if each of the following statements are true or false. If true, give a proof. If false, give a counterexample.

(a) If there are two minimum cost spanning trees $T_1, T_2 \subseteq E$ with $T_1 \neq T_2$, then G contains two edges with the same cost.

(b) If G contains two edges with the same cost, then G has two minimum cost spanning trees $T_1, T_2 \subseteq E$ with $T_1 \neq T_2$.

Problems 2 and 3 on the next page.

Problem 2. Let $G = (V, E)$ be a connected graph on $|V| = n$ vertices. We call $F \subseteq E$ a *k-forest* of G if it is a forest with $|F| = k$ edges.

(a) Show that a k-forest has $n - k$ connected components and that any subgraph (V, F) with $|F| = k$ and $n - k$ connected components is a k-forest.

(b) Show that if F, $\tilde{F}$ are two k-forests in G, then for any $e \in F \backslash \tilde{F}$, there is an edge $\tilde{e} \in \tilde{F} \backslash F$ so that both

$$F \backslash \{e\} \cup \{\tilde{e}\} \text{ and } \tilde{F} \backslash \{\tilde{e}\} \cup \{e\}$$

are forests.

Problem 3. Let $G = (V, E)$ be a connected graph on $|V| = n$ vertices with edge costs $c(e) \in \mathbb{R}_{\geq 0}$ for $e \in E$ and let $k \leq n - 1$. The cost of a forest F is $\text{cost}(F) = \sum_{e \in F} c(e)$.

(a) Show that if F is not a minimum cost k-forest of G, then for some $e \in F$ and $\tilde{e} \in E \backslash F$, $F \backslash \{e\} \cup \{\tilde{e}\}$ is a k-forest with lower cost.

(b) Consider the following variant of Kruskal's algorithm:

(1) Set $F_0 := \emptyset$
(2) Sort the edges so that $c(e_1) \leq c(e_2) \leq \ldots \leq c(e_m)$
(3) For $i = 1$ to m do
    (4) If $F_{i-1} \cup \{e_i\}$ is acyclic and $|F_{i-1} \cup \{e_i\}| \leq k$,
        update $F_i := F_{i-1} \cup \{e_i\}$
        Otherwise take $F_i := F_{i-1}$
(5) Output $F = F_m$

Show that the output is a minimum-cost k-forest in G.

(c) Find a minimum cost 7-forest of the following graph (edge costs in blue).