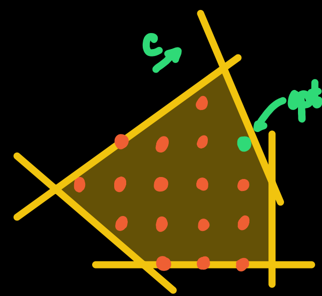
Math 409: Discrete Optimization
Today: Integer Programming

Integer programming

An integer program is a problem of the form

(IP) $\max c^T x$ s.t. $Ax \le b$, $x \in \mathbb{Z}^n$



Remark: Integer Programming is NP-Hard!

For example, we can model MIN VERTEX COVER for arbitrary graphs as an (IP):

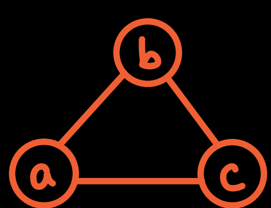
Input: $G=(V,E)$ Goal: $\min |W|$ s.t. $|e \cap W| \ge 1$ $\forall e \in E$

$v \in V \to$ variable y_v, constrained $0 \le y_v \le 1$ ($y_v = 1 \leftrightarrow v \in W$, $y_v = 0 \leftrightarrow v \notin W$)

opt val $= \min \sum_{v \in V} y_v$ s.t. $y \in \mathbb{Z}^V$, $0 \le y_v \le 1$ $\forall v$

$y_u + y_v \ge 1$ for all $\{u,v\} \in E$.

Ex: $G = K_3$



$\min y_a + y_b + y_c$ s.t. $y_a, y_b, y_c \in \{0,1\}$

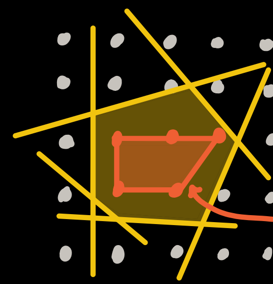
$y_a + y_b \ge 1$, $y_a + y_c \ge 1$, $y_b + y_c \ge 1$

opt val $= 2$, opt sol: $(1,1,0), (1,0,1), (0,1,1) \longleftrightarrow$ min vertex covers

Dropping "$x \in \mathbb{Z}$ constraint $\to$ LP relaxation:

(LP) $\max c^T x$ s.t. $Ax \le b$ ← gives upper bound (IP) $\le$ (LP)

The <u>integer hull</u> of a polyhedron $P=\{x\in\mathbb{R}^n : Ax\le b\}$ is the convex hull of its integer points:



$$P_I = \operatorname{conv}(P\cap\mathbb{Z}^n)$$

$\Longrightarrow$ (IP) $\max c^Tx$ s.t. $x\in P\cap\mathbb{Z}^n$

$= \max c^Tx$ s.t. $x\in P_I$

($\le$) follows from $P\cap\mathbb{Z}^n\subseteq P_I$

($\ge$) $x\in P_I \Rightarrow x=\sum_{i=1}^{k}\lambda_i v_i$ for some $v_1,\dots,v_k\in P\cap\mathbb{Z}^n$, $\lambda_1,\dots,\lambda_k\ge 0$, $\sum\lambda_i=1$

$\Rightarrow c^Tx=\sum\lambda_i c^Tv_i \le \sum\lambda_i\alpha=\alpha$ where $\alpha=$ opt val of (IP)

<u>Prop</u>: For a polytope $P\subseteq\mathbb{R}^n$, $P=P_I$

$\Longleftrightarrow$ for all $c\in\mathbb{R}^n$, $\underbrace{\max\{c^Tx : x\in P\}}_{\text{(LP)}} = \underbrace{\max\{c^Tx : x\in P\cap\mathbb{Z}^n\}}_{\text{(IP)}}$

<u>LP for max matchings in $G=(V,E)$</u>

$\begin{pmatrix} x_e=1 \leftrightarrow e\in M \\ x_e=0 \leftrightarrow e\notin M \end{pmatrix}$

$e\in E \rightarrow$ variable x_e, constrained $0\le x_e\le 1$

Constraints: $\deg_v(M)\le 1 \rightarrow \sum_{e\ni v} x_e\le 1 \quad \forall v\in V$

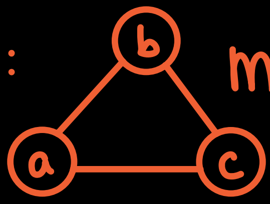
Maximize $|M|$ (or $\sum_{e\in M} c_e$ for some $c: E\to\mathbb{R}$) $\rightarrow \max\sum_{e\in E} c_e x_e$

(IP) $\max\sum_{e\in E} c_e x_e$ s.t. $0\le x_e\le 1$ for all $e\in E$, $x\in\mathbb{Z}^E$

$\sum_{e\ni v} x_e\le 1$ for all $v\in V$

<u>Claim</u>: This solves max weight matching on G

(Recover M from opt sol x^* by $M=\{e : x_e^*=1\}$) You check!

Ex: 

$G = K_3$

$\max\ x_{ab} + x_{ac} + x_{bc}$ s.t. $0 \le x_{ab} \le 1,\ 0 \le x_{ac} \le 1,\ 0 \le x_{bc} \le 1$

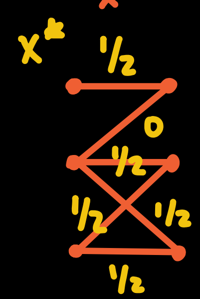
$x_{ab} + x_{ac} \le 1,\ x_{ab} + x_{bc} \le 1,\ x_{ac} + x_{bc} \le 1$

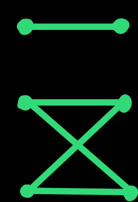
For $x \in \mathbb{Z}^3$, opt val $= 1$, opt sol's : $(1,0,0), (0,1,0), (0,0,1)$

For $x \in \mathbb{R}^3$, opt val $= 3/2$, opt sol : $(1/2, 1/2, 1/2)$

<u>Thm</u>: For a bipartite graphs, any vertex x^* of the matching LP is integer. ($\Rightarrow P = P_I$ for $P =$ feas. region of matching LP)

(Proof) Suppose x^* is a vertex. Define $G_{x^*} = (V, E_{x^*})$ with $E_{x^*} = \{e \in E \text{ s.t. } 0 < x^*_e < 1\}$.

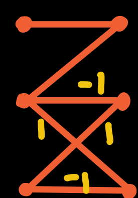




(Case 1) Suppose G_{x^*} has a cycle C

G_{x^*} bipartite $\Rightarrow C$ has even length : $C = (e_1, e_2, \ldots, e_{2k})$

Define $w \in \mathbb{R}^E$ by $w_e = \begin{cases} 1 & \text{if } e = e_{2i} \in C \\ -1 & \text{if } e = e_{2i+1} \in C \\ 0 & \text{o.w.} \end{cases}$



Take $\varepsilon = \min\{x_e : e \in C\} \cup \{1 - x_e : e \in C\}$

Then <u>$x^* + \varepsilon w$ and $x^* - \varepsilon w$ both feasible</u> $\Rightarrow x^*$ not a vertex

check $(x^* \pm \varepsilon w)_e \in [0,1]\ \forall e$ and $\sum_{e \in v} (x^* \pm \varepsilon w)_e = \sum_{e \ni v} x^*_e$