# Math 409: Discrete Optimization

## Today: Linear Programming

## Linear programming (Recall)

A linear program (LP) is a problem of the form

$$\max_{x \in \mathbb{R}^n} \; c^T x \quad \text{s.t.} \quad Ax \le b \qquad \text{where } c \in \mathbb{R}^n,\ A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m$$

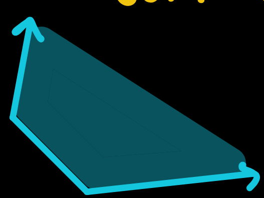
$$c^T x = \sum_{i=1}^{n} c_i x_i \qquad Ax \le b: \begin{array}{c} a_1^T x \le b_1 \\ \vdots \\ a_m^T x \le b_m \end{array} \quad \text{where } A = \begin{pmatrix} - a_1^T - \\ \vdots \\ - a_m^T - \end{pmatrix}$$

The feasible set $P = \{x \in \mathbb{R}^n : Ax \le b\}$ is a polyhedron

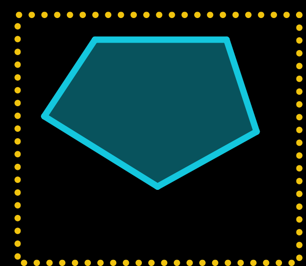
A bounded polyhedron is a polytope

bounded: enclosed in some box $\{x \in \mathbb{R}^n : -r \le x_1 \le r, \dots, -r \le x_n \le r\}$

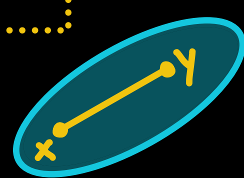
(unbounded) polyhedron



polytope



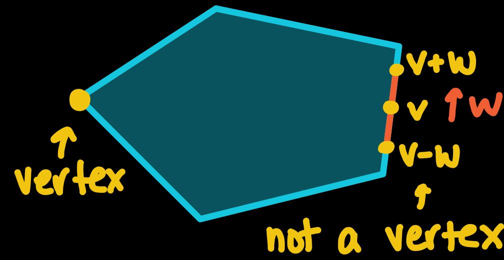
The set $P$ is convex, meaning that $x, y \in P,\ 0 \le \lambda \le 1 \Rightarrow \lambda x + (1-\lambda) y \in P$




The convex hull of pts $\{v_1, \dots, v_t\} \subseteq \mathbb{R}^n$ is

$$\mathrm{Conv}\{v_1, \dots, v_t\} = \Big\{ \underbrace{\sum_{i=1}^{t} \lambda_i v_i}_{\text{a convex combination of } v_1, \dots, v_t} : \lambda_1, \dots, \lambda_t \ge 0,\ \sum_{i=1}^{t} \lambda_i = 1 \Big\}$$

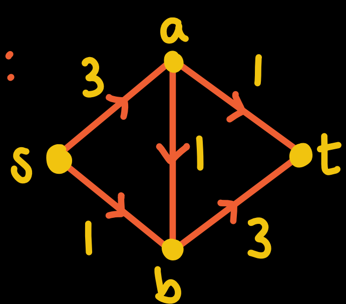
We call $v \in P$ a <u>vertex</u> of $P$ if there is no nonzero $w \in \mathbb{R}^n \setminus \{0\}$ s.t. $v+w \in P$ and $v-w \in P$




<u>Thm</u>: If $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is bounded then $\max\{c^T x : x \in P\}$ is acheived by a vertex $x^* \in P$.
Moreover, if $I = \{i \in [m] : a_i^T x^* = b_i\}$, then

$$\{x \in \mathbb{R}^n : a_i^T x = b_i \ \forall i \in I\} = \{x^*\}$$

Ex:




$$\{\text{s-t flows}\} = \left\{ x \in \mathbb{R}^5 : \begin{array}{l} x_{sa} = x_{ab} + x_{at}, \\ x_{sb} + x_{ab} = x_{bt} \end{array} \quad \begin{array}{l} 0 \leq x_{sa} \leq 3 \\ 0 \leq x_{sb} \leq 1 \\ 0 \leq x_{ab} \leq 1 \\ 0 \leq x_{at} \leq 1 \\ 0 \leq x_{bt} \leq 3 \end{array} \right\}$$

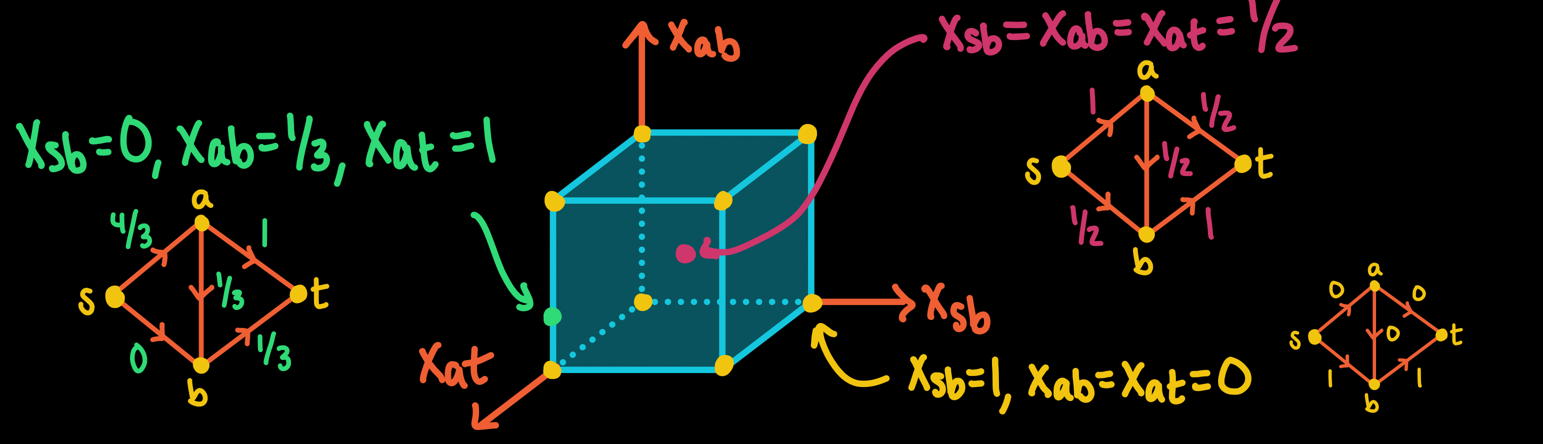
Reformulate as $Ax \leq b$:

$0 \leq x_{sa} \leq 3 \iff x_{sa} \leq 3$ and $-x_{sa} \leq -0$

$x_{sa} = x_{ab} + x_{at} \iff x_{sa} - x_{ab} - x_{at} \leq 0$ and $-x_{sa} + x_{ab} + x_{at} \leq 0$

In total: $m = 14$ ineq. in $n = 5$ variables

Feasible set $\cong$ 0-1 cube in $\mathbb{R}^3$



# Duality in Linear Programming

(P) $\max\ c^T x$ s.t. $Ax \le b$

$$a_1^T x \le b_1 \quad (y_1)$$
$$\vdots \qquad\qquad \vdots$$
$$a_m^T x \le b_m \quad (y_m)$$

We look for upper bounds on $c^T x$ coming from nonnegative linear combinations of constraints:

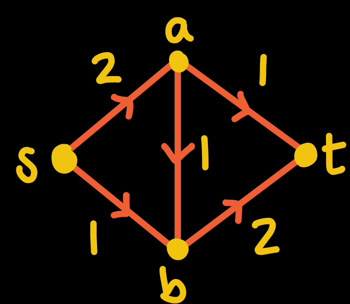
$$y_1(a_1^T x) + \dots + y_m(a_m^T x) \le y_1 b_1 + \dots + y_m b_m \qquad (y_i \ge 0)$$

Best (= lowest) upper bound on $c^T x$? $\to$ Dual LP

(D) $\min\ b^T y$ s.t. $y \ge 0$, $y^T A = c^T$

Ex from above: $c^T x = x_{sa} + x_{sb}$

opt val. for (P) = 3, acheived by




Certify using dual LP?

$$\begin{aligned}
1\,(\qquad\qquad\qquad\quad x_{sb} &\le 1) \\
+1\,(\qquad\qquad\qquad\quad x_{ab} &\le 1) \\
+1\,(\qquad\qquad\qquad\quad x_{at} &\le 1) \\
+1\,(x_{sa} - x_{ab} - x_{at} &\le 0) \\
\hline
x_{sa} + x_{sb} &\le 3
\end{aligned}$$

$\leftarrow$ 4 dual variables $y_i$ take value 1, other 10 take value 0

<u>Thm (Weak Duality)</u> If $x, y$ are feasible for (P), (D) (resp.) then $c^T x \le b^T y$. $\Rightarrow$ (P) $\le$ (D)

(Proof) $x$ feas. $\iff Ax \le b$, $y$ feas $\iff y \ge 0$, $y^T A = c^T$

$$\Rightarrow c^T x = y^T A x = \sum_{i=1}^{m} y_i a_i^T x \underset{\substack{\uparrow \\ y_i \ge 0,\ a_i^T x \le b_i}}{\le} \sum_{i=1}^{m} y_i b_i = b^T y$$

Note: For equality '$\le$', need $\underbrace{y_i(b_i - a_i^T x) = 0 \quad \forall i}_{\text{"complimentary slackness"}}$

**Thm (Strong Duality)** If (P) and (D) are both feasible, then (P)=(D). That is, $\exists x^* \in \mathbb{R}^n$, $y^* \in \mathbb{R}^m$ s.t. $Ax^* \le b$, $y^* \ge 0$, $(y^*)^T A = c^T$, $c^T x^* = b^T y^*$.

**Thm (Complimentary Slackness)**

Suppose $x, y$ are feasible for (P), (D) resp.

Then both are optimal

$\iff c^T x = b^T y$

$\iff$ for all $i = 1, \dots, m$, $y_i(a_i^T x - b_i) = 0$.