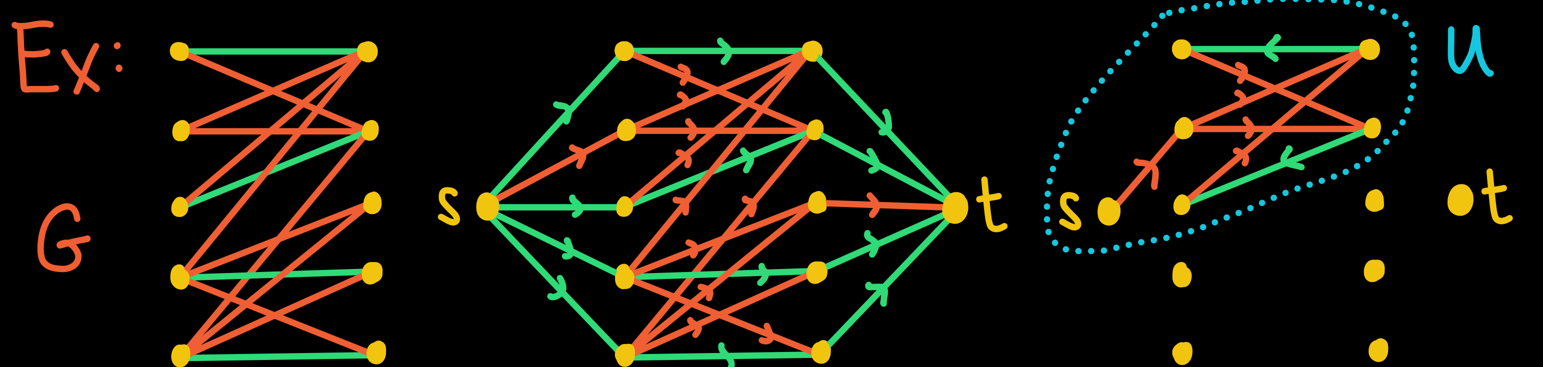
Math 409: Discrete Optimization

Today: Sum up of bipartite graph applications
Start connections with LPs

Recall: $G = (V_1 \cup V_2, E)$ bipartite graph
$\rightarrow G' = (V_1 \cup V_2 \cup \{s,t\}, E')$ auxiliary digraph
max s-t flow $f: E' \to \mathbb{Z}_{\geq 0}$, min s-t cut U
gives max matching $M = \{\{v,w\} \in E : f(v,w) = 1\}$
and min vertex cover $W = (V_1 \setminus U) \cup (V_2 \cap U)$
with same size $|M| = |W|$ (König's Thm)

If $|V_1| = |V_2| = n$ and $|M| < n$, then (Hall's Thm)
$\tilde{U} = V_1 \cap U = V_1 \setminus W$ satisfies $|N(\tilde{U})| < |\tilde{U}|$.

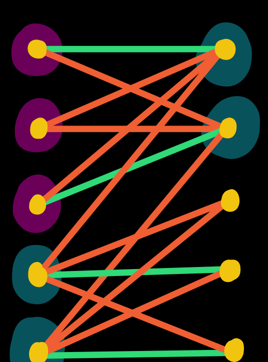
Ex:



matching $|M| = 4$

$\longrightarrow$ $f(e) = 1$
$\longrightarrow$ $f(e) = 0$
flow in G'

$U = \{ v$ s.t. $\exists$ s-v path in residual graph $G'_f \}$



$W = (V_1 \setminus U) \cup (V_2 \cap U)$ vertex cover with $|W| = 4$

$\tilde{U} = V_1 \cap U = V_1 \setminus W$ has $|N(\tilde{U})| < |\tilde{U}|$

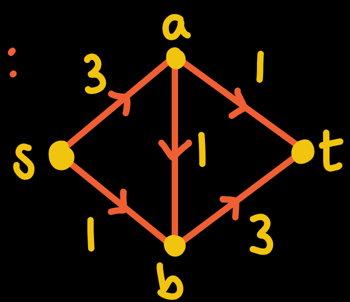
Connections to Linear Programming

Given digraph $G=(V,E)$, $c: E \to \mathbb{Z}_{\geq 0}$, $s,t \in V$, the set of s-t flows $f: E \to \mathbb{R}_{\geq 0}$ is defined by linear equations and inequalities in $x_e = f(e)$:

$0 \leq x_e \leq c(e)$ for all $e \in E$ and $\displaystyle\sum_{e \in \delta^{in}(v)} x_e = \sum_{e \in \delta^{out}(v)} x_e \quad \forall v \in V \setminus \{s,t\}$

Over this set, we want to maximize the linear function: $\displaystyle\sum_{e \in \delta^{out}(s)} x_e - \sum_{e \in \delta^{in}(s)} x_e$

This is a linear program!

Ex: 

Max value s-t flow given by

$\max \; x_{sa} + x_{sb}$ s.t.

$x_{sa} = x_{ab} + x_{at}$ $0 \leq x_{sa} \leq 3$

$x_{sb} + x_{ab} = x_{bt}$ $0 \leq x_{sb} \leq 1$

$0 \leq x_{ab} \leq 1$

$0 \leq x_{at} \leq 1$

$0 \leq x_{bt} \leq 3$

Max value $= 3$ given by

$(x_{sa}, x_{sb}, x_{ab}, x_{at}, x_{bt}) = (2, 1, 1, 1, 2)$

Upper bound: $3 - x_{sa} - x_{sb} =$

$$\underbrace{(1 - x_{sb})}_{\geq 0} + \underbrace{(1 - x_{ab})}_{\geq 0} + \underbrace{(1 - x_{at})}_{\geq 0} + \underbrace{(x_{ab} + x_{at} - x_{sa})}_{=0}$$

Parametrize by (x_{sb}, x_{ab}, x_{at})

$\to 0 \le x_{sa} = \underline{x_{ab} + x_{at} \le 3}$, $\quad 0 \le x_{bt} = \underline{x_{sb} + x_{ab} \le 3}$

implied by $x_{ab} \le 1$, $x_{at} \le 1$, $x_{sb} \le 1$

Feasible set $\cong$ 0-1 cube in $\mathbb{R}^3$