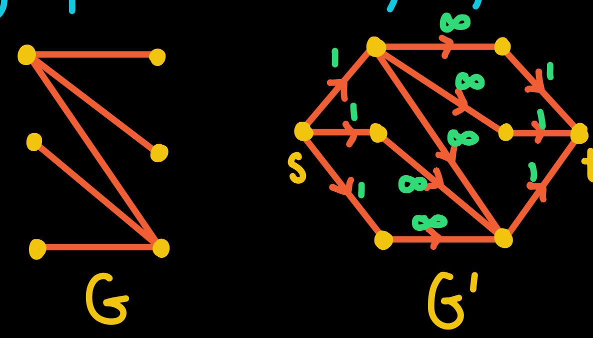
Math 409: Discrete Optimization

Today: Applications to bipartite graphs, cont'

Midterm in class Wed May 1

From last time... given bipartite graph $G=(V_1 \cup V_2, E)$, create digraph $G'=(V_1 \cup V_2, E')$

$E' = \{(s,v) : v \in V_1\} \cup \{(w,t) : w \in V_2\}$
$\{(v,w) : \{v,w\} \in E, v \in V_1\}$



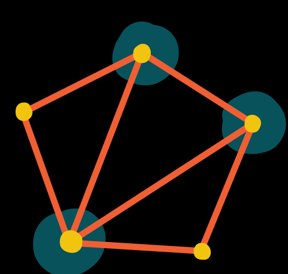
capacities $c(s,v) = c(w,t) = 1$
and $c(v,w) = \infty$

← Note: for all proofs it suffices to take $c(v,w) \geq |V_1|+1$
$\Rightarrow$ (v,w) can't appear in $\delta^{out}(U)$ for min cut U

<u>Prop</u>: size of max matching in G
= max value of s-t flow in G'

<u>Vertex Covers</u>

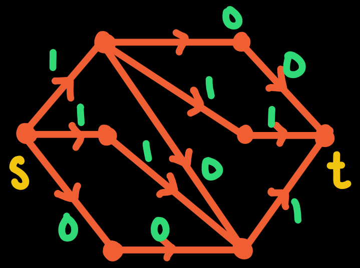
$U \subseteq V$ is a <u>vertex cover</u> of $G=(V,E)$ if $|e \cap U| \geq 1$ for all $e \in E$



<u>König's Thm</u>: In a bipartite graph G, max size of matching = min size vertex cover

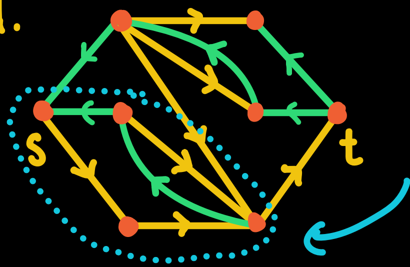
(Proof) Take max s-t flow $f: E' \to \mathbb{Z}_{\geq 0}$ in G' and corresponding min cut $U \subseteq V \cup \{s,t\}$ with $c(\delta^{out}(U)) = \text{value}(f) = |M|$ for max size matching.

G'



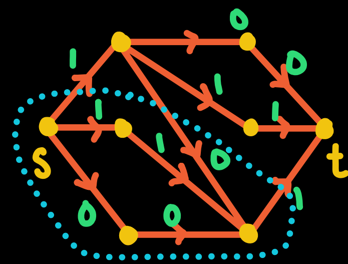
flow f

resid. G'_f

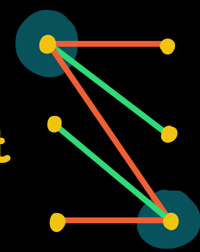


min cut $U = \{v \in V \text{ reachable from } s \text{ in } G'_f\}$

Claim: $W = (V_1 \setminus U) \cup (V_2 \cap U)$ is a vertex cover with $|W| = |M|$



U



W

(Size) $v \in V_1 \setminus U \iff (s,v) \in \delta^{out}(U)$

$w \in V_2 \cap U \iff (w,t) \in \delta^{out}(U)$

Since $c(v,w) = \infty$ for $v \in V_1, w \in V_2$ and $c(\delta^{out}(U)) \le |V_1|$, there are no edges (v,w) in $\delta^{out}(U)$.

$$\Rightarrow |M| = c(\delta^{out}(U)) = \sum_{v \in V_1 \setminus U} c(s,v) + \sum_{w \in V_2 \cap U} c(w,t) = |W|$$

(Vertex cover) Let $\{v,w\} \in E$ with $v \in V_1, w \in V_2$.

$c(v,w) = \infty \Rightarrow (v,w) \notin \delta^{out}(U) \Rightarrow v \notin U$ or $w \in U \Rightarrow v \in W$ or $w \in W$.

<u>Cor</u>: This gives an algorithm to find a min vertex cover of a bipartite graph in time $O(n \cdot m)$.

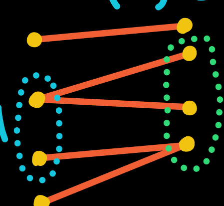
$n = |V| \quad m = |E|$

<u>Another nice consequence</u>: Hall's Thm

The <u>neighborhood</u> of $U \subseteq V$ in a graph $G = (V,E)$ is

$N(U) = \{w \in V : \{v,w\} \in E \text{ for some } v \in U\}$

U

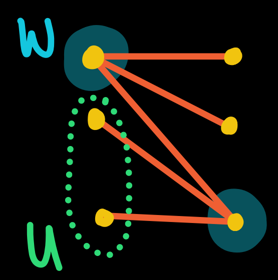


$N(U)$

<u>Hall's Thm</u> A bipartite graph $G=(V_1 \cup V_2, E)$ with $|V_1|=|V_2|=n$ has a perfect matching $(|M|=n)$ if and only if $|N(U)| \geq |U|$ for all $U \subseteq V_1$.

(Proof) ($\Rightarrow$) perfect matching gives injection $U \to N(U)$

($\Leftarrow$) Suppose max matching has size $\leq n-1$



$\overset{\Longrightarrow}{\text{König}}$ $\exists$ vertex cover $W \subseteq V$ of size $\leq n-1$

Take $U = V_1 \setminus W$.

Any edge $\{v,w\}$ with $v \in U$ is covered by $W \Rightarrow w \in W \cap V_2$

$\Rightarrow N(U) \subseteq W \cap V_2$

Note that $n = |V_1| = |U| + |V_1 \cap W|$ since $V_1 = U \uplus (V_1 \cap W)$

and $|V_1 \cap W| + |V_2 \cap W| = |W| \leq n-1$ since $W = (V_1 \cap W) \uplus (V_2 \cap W)$

$\Rightarrow N(U) \leq |V_2 \cap W| \leq (n-1) - |V_1 \cap W| = (n - |V_1 \cap W|) - 1 = |U| - 1$