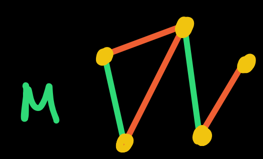
Math 409: Discrete Optimization

Today: Matchings in bipartite graphs

Recall:

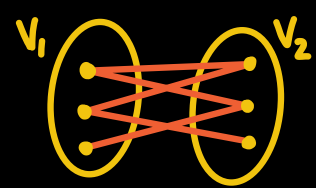
A <u>matching</u> of $G=(V,E)$ is a set $M \subseteq E$ in which every vertex has $\deg \le 1$.



<u>MAX CARDINALITY MATCHING</u>
Input: graph $G=(V,E)$
Goal: Find matching $M \subseteq E$ of maximum size $\lvert M \rvert$

Today we focus on a special case:

A graph $G=(V,E)$ is <u>bipartite</u> if $V = V_1 \cup V_2$ and $\{v,w\} \notin E$ for $\{v,w\} \subseteq V_i$

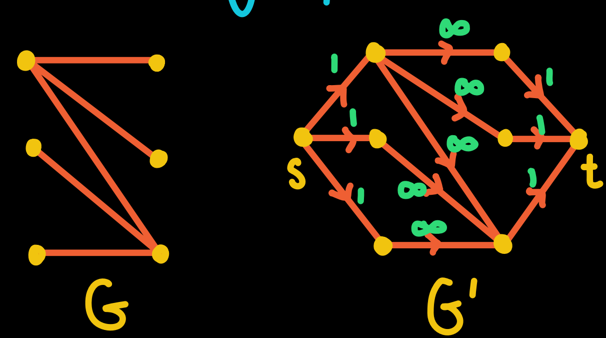


From $G=(V_1 \cup V_2, E)$, define directed graph G'

vertices: $\{s,t\} \cup V_1 \cup V_2$

edges: $\{(s,v) : v \in V_1\} \cup \{(w,t) : w \in V_2\}$

E' $\quad \cup \{(v,w) : \{v,w\} \in E, v \in V_1, w \in V_2\}$

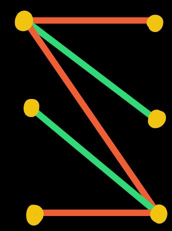


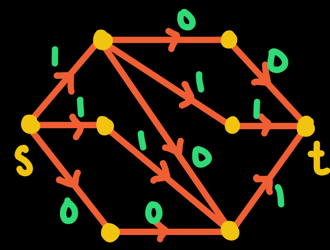
Define $c : E' \to \mathbb{Z}_{\ge 0}$ by $c(s,v) = c(w,t) = 1$ and $c(v,w) = \infty$

<u>Prop</u>: size of max matching in G

$= $ max value of s-t flow in G'

(Proof) ($\leq$) Given matching $M \subseteq E$, define flow $f: E' \to \mathbb{Z}_{\geq 0}$





$$f(e) = \begin{cases} 1 & \text{if } (s,v),\ v \text{ matched in } M, \\ 1 & \text{if } (v,w) \text{ for } \{v,w\} \in M \\ 1 & \text{if } (w,t) \text{ for } w \text{ matched in } M \\ 0 & \text{otherwise} \end{cases}$$

Then f is an s-t flow of value $|M|$

($\geq$) Let f be the output of the Ford-Fulkerson Alg.

$\Rightarrow f$ is max value flow and $f(e) \in \mathbb{Z}_{\geq 0}$ for all $e \in E$.

Take $M = \{\{v,w\} \in E : f(v,w) = 1\}$.

Since $f(s,v) \leq 1 \ \forall v \in V_1$, $\sum_{e \in \delta^{in}(v)} f(e) = f(s,v) = \sum_{e \in \delta^{out}(v)} f(e) \leq 1$

$\Rightarrow f(v,w) = 1$ for at most one $w \in V_2$ $\Rightarrow M$ is a matching

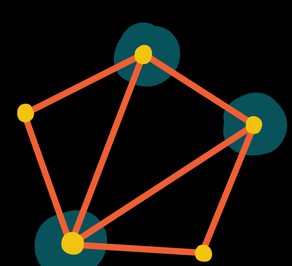
with $\text{value}(f) = \sum_{e \in \delta^{out}(s)} f(e) = |M|$

Cor: This gives an algorithm to find a max matching of a bipartite graph in time $O(n \cdot m)$.

$n = |V| \quad m = |E|$

Vertex Covers

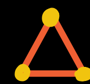
$\mathcal{U} \subseteq V$ is a vertex cover of $G = (V,E)$ if $|e \cap \mathcal{U}| \geq 1$ for all $e \in E$



MIN VERTEX COVER

Input: $G=(V,E)$ **Output:** min size vertex cover of G

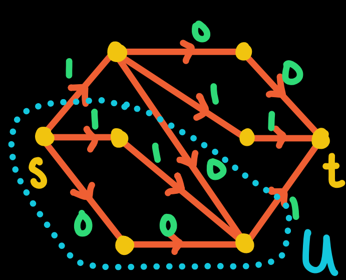
König's Thm: In a bipartite graph G,
max size of matching = min size vertex cover

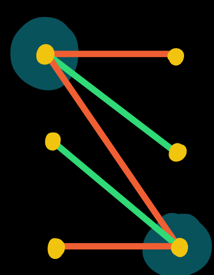
("$\leq$" holds in any graph, but not "$=$", e.g. $G=K_3$ )

Proof next time. Idea:

Take max s-t flow $f: E' \to \mathbb{Z}_{\geq 0}$ in G'
and corresponding min cut $U \subseteq V \cup \{s,t\}$ with
$c(\delta^{out}(U)) = \text{value}(f) = |M|$.

Claim: $W = (V_1 \setminus U) \cup (V_2 \cap U)$
is a vertex cover of size $|W| = |M|$.

Ex: 



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