Math 409: Discrete Optimization

Today: Min cut / Max Flow

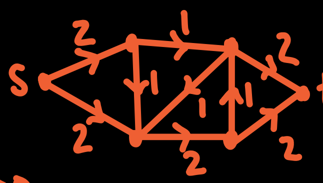
Network Flows

$G=(V,E)$ directed graph, $c: E \to \mathbb{Z}_{\geq 0}$ edge capacities

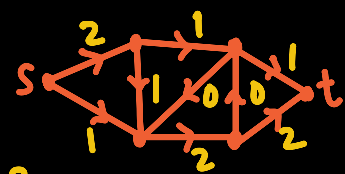
$s, t \in V$ $s =$ "source" $t =$ "sink"

Last time: find max s-t flow in G w/ augmenting paths

Ex:



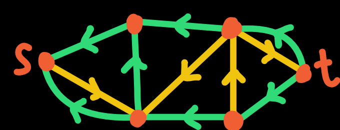
$c: E \to \mathbb{Z}_{\geq 0}$



value(f) = 3

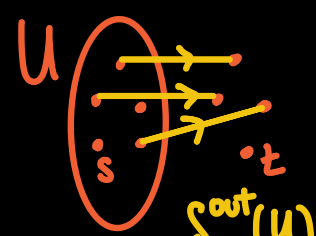
$f: E \to \mathbb{R}_{\geq 0}$

Can we do better?



no s-t path in res. graph G_f

An s-t cut in G is a subset $U \subseteq V$ of vertices with $s \in U$ and $t \notin U$.

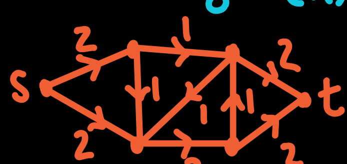


$\delta^{out}(U)$

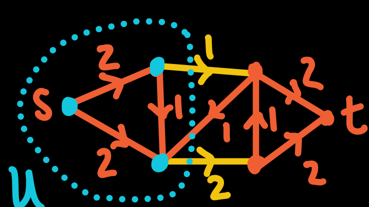
The capacity of an s-t cut $U \subseteq V$ is the total capacity of edges leaving U:

$$c(\delta^{out}(U)) = \sum_{e \in \delta^{out}(U)} c(e) \quad \text{where } \delta^{out}(U) = \{(i,j) \in E : i \in U, j \notin U\}$$

Ex:



$c: E \to \mathbb{Z}_{\geq 0}$



$c(\delta^{out}(U)) = 1 + 2 = 3$

MIN CUT PROBLEM

Input: $G=(V,E)$, $c: E \to \mathbb{Z}_{\geq 0}$, $s, t \in V$

Goal: Find s-t cut $U \subseteq V$ with minimum cap. $c(\delta^{out}(U))$

Prop 1: For any s-t flow $f: E \to \mathbb{Z}_{\geq 0}$ and s-t cut $U \subseteq V$

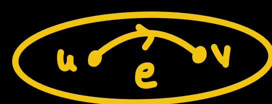
$$\text{value}(f) \leq c(\delta^{out}(U))$$

(Proof)

$$\text{value}(f) = \sum_{e \in \delta^{out}(s)} f(e) - \sum_{e \in \delta^{in}(s)} f(e) + \sum_{v \in U \setminus \{s\}} \Big(\overbrace{\sum_{e \in \delta^{out}(v)} f(e) - \sum_{e \in \delta^{in}(v)} f(e)}^{=0} \Big)$$

$$= \sum_{v \in U} \Big(\sum_{e \in \delta^{out}(v)} f(e) - \sum_{e \in \delta^{in}(v)} f(e) \Big)$$

$$= \sum_{e \in \delta^{out}(U)} f(e) - \sum_{e \in \delta^{in}(U)} f(e)$$



$e \in \delta^{out}(u)$ and $\delta^{in}(v)$

edges in U cancel out

$$\leq \sum_{e \in \delta^{out}(U)} f(e) \leq \sum_{e \in \delta^{out}(U)} c(e) = c(\delta^{out}(U))$$

Note: $\text{value}(f) = c(\delta^{out}(U)) \iff$ both "$\leq$" above are "$=$"

$\iff$ $f(e) = 0$ for all $e \in \delta^{in}(U)$

and $f(e) = c(e)$ for all $e \in \delta^{out}(U)$.

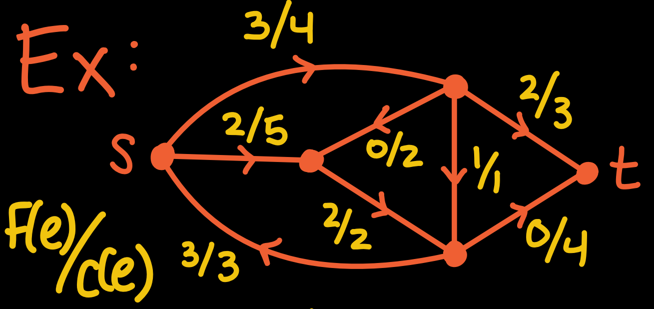
Recall: Given s-t flow $f: E \to \mathbb{Z}_{\geq 0}$, residual graph G_f has edges (v,w) when $f(v,w) < c(v,w)$

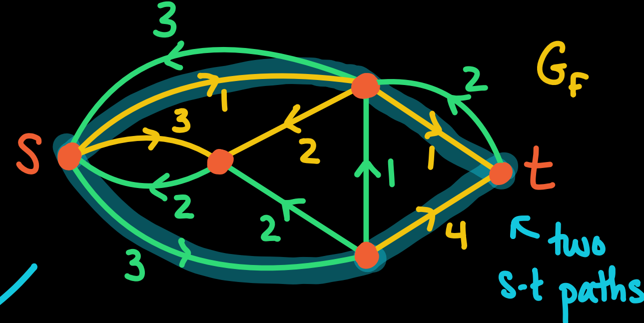
(w,v) when $f(v,w) > 0$

s-t paths in G_f give f-augmenting paths

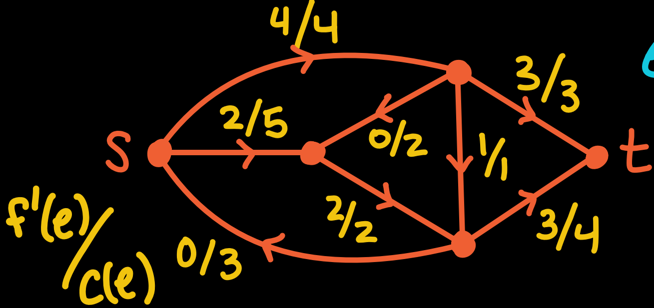
$\to$ get an s-t flow of greater value

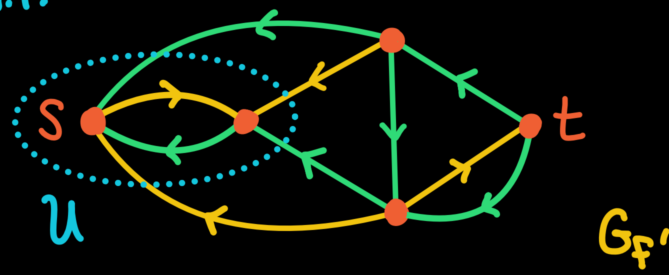
Ex:





aug. w.r.t. both





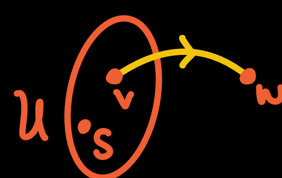
<u>Prop 2</u>: If f is an s-t flow and there is no s-t path in G_f then

$$U = \{v \in V \text{ s.t. there is an } s\text{-}v \text{ path in } G_f\}$$

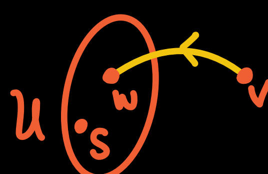
is an s-t cut with $c(\delta^{out}(U)) = \text{value}(f)$.

(Proof) Since $\nexists$ s-t path in G_f, $t \notin U$. Since $s \in U$, U is an s-t cut. To show $\text{value}(f) = c(\delta^{out}(U))$ it suffices to show that $f(e) = 0 \ \forall e \in \delta^{in}(U)$, $f(e) = c(e) \ \forall e \in \delta^{out}(U)$

- $e = (v,w) \in \delta^{out}(U) \Rightarrow v \in U, w \notin U$
  $\Rightarrow \exists$ s-v path in G_f, $\nexists$ s-w path in G_f
  $\Rightarrow$ e is not an edge in $G_f \Rightarrow f(e) = c(e)$



- $e = (v,w) \in \delta^{in}(U) \Rightarrow v \notin U, w \in U$
  $\Rightarrow \exists$ s-w path in G_f, $\nexists$ s-v path in G_f
  $\Rightarrow e^{-1} = (w,v)$ is not an edge in $G_f \Rightarrow f(e) = 0$



Together Props 1 and 2 give:

Max Flow - Min Cut Thm:

In any network, the max value of an s-t flow equals the min capacity of an s-t cut.

(Proof) Take f to be a max value flow.

$\xRightarrow{\text{last time}}$ G_f has no s-t path (would give higher value flow)

$\xRightarrow{\text{Prop 2}}$ $\exists$ s-t cut U with $\text{value}(f) = c(\delta^{out}(U))$

$\xRightarrow{\text{Prop 1}}$ U is a min capacity cut