# Math 409: Discrete Optimization

Today: Network Flows

## Network Flows

$G=(V,E)$ directed graph, $c: E \to \mathbb{Z}_{\geq 0}$ edge capacities

$s,t \in V$ $\quad s=$ "source" $\quad t=$ "sink"

A function $f: E \to \mathbb{R}_{\geq 0}$ is called an <u>s-t flow</u>

if $0 \leq f(e) \leq c(e)$ for all $e \in E$ and

$$\sum_{e \in \delta^{in}(v)} f(e) = \sum_{e \in \delta^{out}(v)} f(e) \text{ for all } v \in V \setminus \{s,t\}$$

conservation of flow at v

flow into v = flow out of v

where $\delta^{in}(v) = \{(w,v) \in E\} = \{\text{edges entering } v\}$

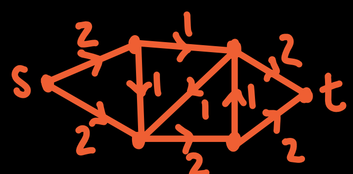
$\delta^{out}(v) = \{(v,w) \in E\} = \{\text{edges leaving } v\}$

The <u>value</u> of an s-t flow $f: E \to \mathbb{R}_{\geq 0}$ is

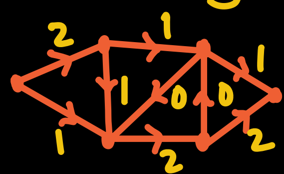
$$\text{value}(f) = \sum_{e \in \delta^{out}(s)} f(e) - \sum_{e \in \delta^{in}(s)} f(e) = \sum_{e \in \delta^{in}(t)} f(e) - \sum_{e \in \delta^{out}(t)} f(e)$$

= net flow leaving s (= net flow entering t)

Ex:




$c: E \to \mathbb{Z}_{\geq 0}$



$f: E \to \mathbb{R}_{\geq 0}$

value(f) = 3

**MAX FLOW PROBLEM**

Input: digraph $G=(V,E)$, $s,t \in V$, $c: E \to \mathbb{Z}_{\geq 0}$

Goal: Find s-t flow of maximum value

## Application (Transportation problem)

$m$ factories, $n$ customers

Factory $f_i$ can produce $p_i$ tons of product/month

Customer $c_j$ wants $d_j$ tons of product/month

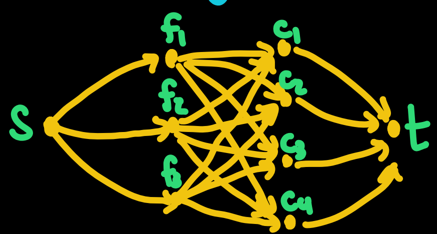
Can transport $t_{ij}$ tons/month from factory $f_i$ to customer $c_j$.

Can product demand be met?

Let $G=(V,E)$ with $V=\{s,t\}\cup\{f_1,\dots,f_m\}\cup\{c_1,\dots,c_n\}$

Edges $E$: $s\to f_i$, $f_i\to c_j$, $c_j\to t$ for $i=1,\dots,m$, $j=1,\dots,n$

Capacity $c(s\to f_i)=p_i$, $c(f_i\to c_j)=t_{ij}$, $c(c_j\to t)=d_j$

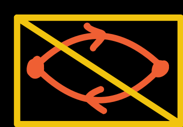



Customer demand can be met

$\iff \exists$ $s$-$t$ flow under $c$ of value $\sum_{j=1}^{n} d_j$

## Flow augmenting paths

Assumption: if $(v,w)\in E$, then $(w,v)\notin E$



Given $s$-$t$ flow $f: E\to\mathbb{R}_{\geq 0}$ define residual graph

$G_f=(V,E_f)$ where $E_f=E_f^+\cup E_f^-$

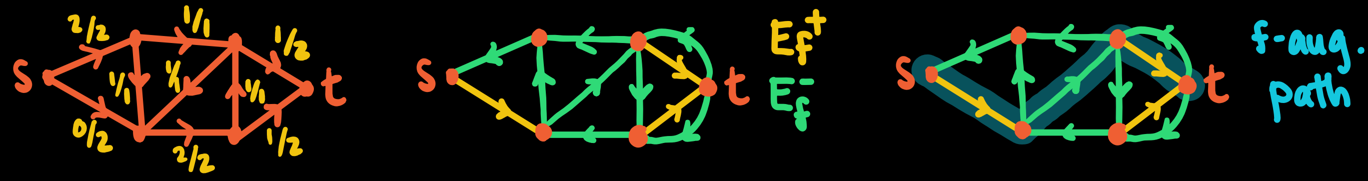
$E_f^+=\{e\in E : f(e)<c(e)\}$ and $E_f^-=\{e^{-1} : e\in E, 0<f(e)\}$

reversed $(v,w)^{-1}=(w,v)$

and residual capacity $c: E_f\to\mathbb{R}_{\geq 0}$

$$c_f(e)=\begin{cases} c(e)-f(e) & \text{if } e\in E_f^+ \\ f(e) & \text{if } e^{-1}\in E_f^- \end{cases}$$

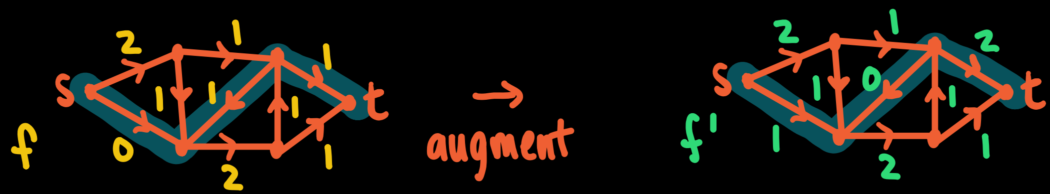
An $s$-$t$ path in $G_f$ is an $f$ augmenting path.

Ex: 


**Thm:** Let $P$ be an $f$-augmenting path with $c_f(e) \geq \gamma$ for all $e \in P$. Define augmented flow $f': E \to \mathbb{R}_{\geq 0}$

$$f'(e) = \begin{cases} f(e) + \gamma & \text{if } e \in P \cap E_f^+ \\ f(e) - \gamma & \text{if } e^{-1} \in P \cap E_f^- \\ f(e) & \text{o.w.} \end{cases}$$

Then $f'$ is an s-t flow with value$(f')$ = value$(f) + \gamma$.

In ex above: 


(Proof) Claim 1: $0 \leq f'(e) \leq c(e)$ for all $e \in E$

- $e \in P \cap E_f^+ \Rightarrow f'(e) = f(e) + \gamma \geq f(e) \geq 0$ and
  $f'(e) = f(e) + \gamma \leq f(e) + c_f(e) = f(e) + (c(e) - f(e)) = c(e)$ ✓
- $e^{-1} \in P \cap E_f^- \Rightarrow f'(e) = f(e) - \gamma \leq f(e) \leq c(e)$ and
  $f'(e) = f(e) - \gamma \geq f(e) - c_f(e) = f(e) - f(e) = 0$
- o.w. $f'(e) = f(e) \Rightarrow 0 \leq f'(e) \leq f(e)$

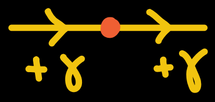
Claim 2: $f'$ is an s-t flow

Take $v \in V \setminus \{s,t\}$. If $v \notin P$, flow in/out of $v$ unchanged.

$v \in P \Rightarrow e_i = (u_{i-1}, v)$, $e_{i+1} = (v, u_{i+1})$ in $P$

Cases:

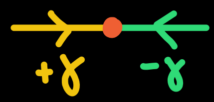
$e_i, e_{i+1} \in E_f^+$




$\delta^{in}(v) \to +\gamma$

$\delta^{out}(v) \to +\gamma$

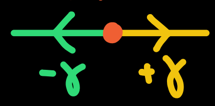
$e_i \in E_f^+, e_{i+1} \in E_f^-$




$\delta^{in}(v) \to +\gamma - \gamma$

$\delta^{out}(v)$ same

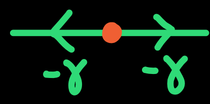
$e_i \in E_f^-, e_{i+1} \in E_f^+$




$\delta^{in}(v)$ same

$\delta^{out}(v) \to -\gamma + \gamma$

$e_i, e_{i+1} \in E_f^-$



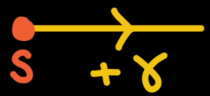

$\delta^{in}(v) \to -\gamma$

$\delta^{out}(v) \to -\gamma$

## Claim 3: $\text{value}(f') = \text{value}(f) + \gamma$

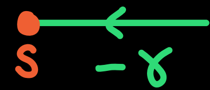
$e_1 = (s, v_1) \in P$ Cases:

$e_1 \in E_f^+$




$\delta^{out}(s) \to +\gamma$ $\delta^{in}(s)$ same

$e_1 \in E_f^-$




$\delta^{out}(s)$ same $\delta^{in}(s) \to -\gamma$