Math 409: Discrete Optimization
Today: Negative cycles

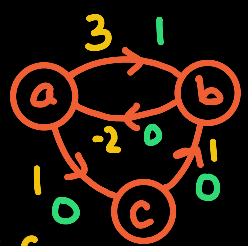
$G=(V,E)$ digraph, $c: E \to \mathbb{R}$

A function $\pi: V \to \mathbb{R}$ is called a <u>feasible potential</u>

if $\underbrace{c(i,j)+\pi(i)-\pi(j)}_{:= c_\pi(i,j),}\geq 0$ for all $(i,j)\in E$

$:= c_\pi(i,j)$, the <u>reduced cost</u> w.r.t. π

Ex:



costs c
costs c_π

$\pi(a)=-2$
$\pi(b)=0$
$\pi(c)=-1$

$c_\pi(a,b)=c(a,b)+\pi(a)-\pi(b)=3-2-0=1\geq 0$

$c_\pi(b,a)=c(b,a)+\pi(b)-\pi(a)=-2+0+2=0$

$c_\pi(a,c)=c(a,c)+\pi(a)-\pi(c)=1-2+1=0$

$c_\pi(c,b)=c(c,b)+\pi(c)-\pi(b)=1-1-0=0$

<u>Thm</u>: G has no negative cycles

$\iff$ there is a function $\pi: V\to\mathbb{R}$ such that

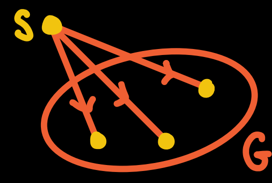
$$c(i,j)+\pi(i)-\pi(j)\geq 0 \text{ for all } (i,j)\in E$$

$(\Leftarrow)$ For any cycle $C: v_1\to v_2\to\dots\to v_k\to v_{k+1}=v_1$

$$\text{cost}(C)=\sum_{j=1}^{k} c(v_j,v_{j+1})+\sum_{j=1}^{k}\pi(v_j)-\sum_{j=1}^{k}\pi(v_j)$$

$$=\sum_{j=1}^{k}\underbrace{c(v_j,v_{j+1})+\pi(v_j)-\pi(v_{j+1})}_{\text{each }\geq 0}\geq 0$$

($\Rightarrow$) Define new graph $\tilde{G}$ by adding vertex s to G and edges (s,v) $\forall v \in V$ with cost 0



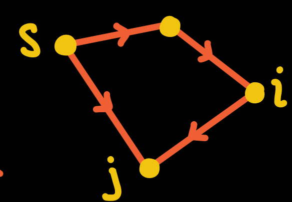
G has no negative cycles $\Rightarrow$ $\tilde{G}$ has no negative cycles

Take $\pi(i) := d(s,i) =$ length of shortest s-i path in $\tilde{G}$

same as shortest s-i walk in $\tilde{G}$

Claim: π is a feasible potential

For $(i,j) \in E$, we can append (i,j) to shortest $s \to i$ walk to get $s \to j$ walk

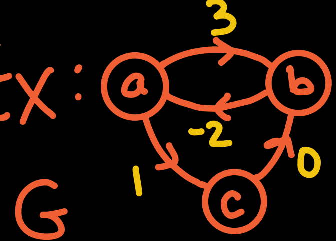


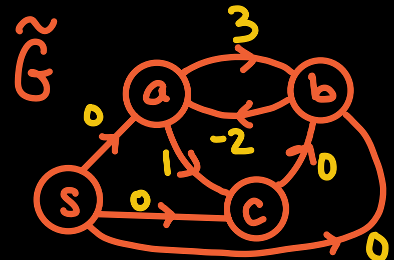
$\Rightarrow \pi(j) = d(s,j) \leq d(s,i) + c(i,j) = \pi(i) + c(i,j)$

$\Rightarrow c(i,j) + \pi(i) - \pi(j) \geq 0$ $\square$

Test for negative cycles by running Moore-Bellman-Ford on $\tilde{G}$ with source s with one extra iteration ($|V|+1$)

Ex:





very abridged:

	e	$\ell(s)$	$\ell(a)$	$\ell(b)$	$\ell(c)$
i=1	⋮	0	0	0	0
i=2	(b,a)	0	-2	0	0
i=3	(a,c)	0	-2	0	-1
i=4	(c,b)	0	-2	-1	0

updated in i=4

Run 4 iterations on $\tilde{G}$

No edge updated in last iteration

$\Rightarrow \ell(w) \leq \ell(v) + c(v,w)$ $\forall (v,w) \in E$ $\Rightarrow \pi(v) = \ell(v)$ feasible potential

$\Rightarrow$ no neg. cycles

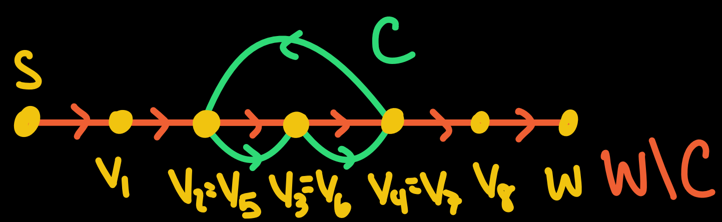
Claim: If $\ell(w)$ is updated in k^{th} iteration (say $\ell(w) \to \ell'(w) = \ell(v) + c(v,w)$), then $\ell'(w)$ is the length of some s-w walk of length $\ell(w)$ with $\geq k$ edges

Why? $\ell(v)$ must have been updated since consideration of (v,w) in $(k-1)^{st}$ iteration (induct on k)

$\ell(w)$ updated extra iteration of alg. $\ell(w) \to \ell(v) + c(v,w)$

$\Rightarrow \ell'(w)$ = length of s-w walk with $\geq n$ edges ($n = \#$vert of $\tilde{G}$)

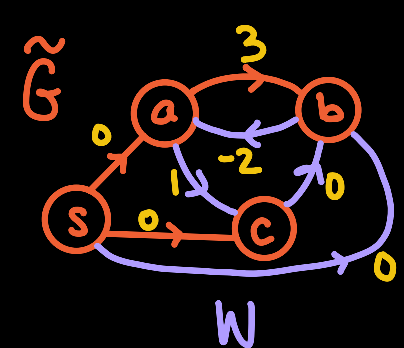
$\Rightarrow$ walk W must have some directed cycle C



If $\text{cost}(C) \geq 0$, then $\text{cost}(W \setminus C) \leq \text{cost}(C)$

$\Rightarrow$ before n^{th} iteration $\ell(w) \leq \text{cost}(W \setminus C) \Rightarrow$ no update

By storing optimal s-v walks along w/ length $\ell(v)$ we find negative cost cycle C as subwalk of W



Thm: This algorithm runs in $O(nm)$ and outputs either a feasible potential or negative cycle.

Running Moore-Bellman Ford takes $O(nm)$.

If no update in last iter. $\Rightarrow$ output $\pi(i) = l(i)$

If $l(w)$ updated, find first return vertex in walk assoc. to w (takes $O(n)$) $\to$ neg. cycle.

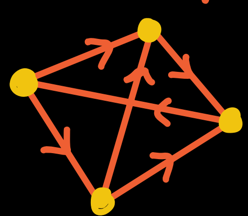
Remark: It is NP-Hard to find min cost paths in arbitrary digraphs w/neg. cycles.

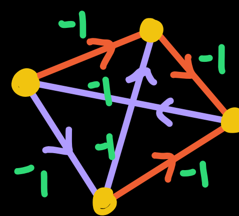
Why? It is NP-Hard to find Hamiltonian paths (vistit every vertex exactly once)

Reduction: Given $G=(V,E)$ define $c(e) = -1 \ \forall e \in E$

$\exists$ Hamiltonian path in $G \Longleftrightarrow$ min cost of path $= n-1$

G





min cost path = -3

$\Rightarrow$ Ham. path