Math 409: Discrete Optimization
Today: Shortest paths w/negative costs

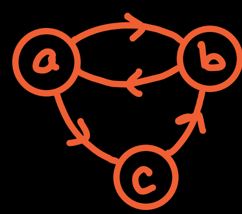
<u>Application</u>: Exchange rates & arbitrage

n types of product (or currency)

x_{ij} = amt of j that can be attained from 1 unit of i

What's the best way to convert product 1 to 2?

Ex: trading apples (a), bananas (b), carrots (c)



You find vendors that will trade

1 apple → 2 bananas $x_{ab}=2$ 1 apple → 6 carrots $x_{ac}=6$

5 bananas → 1 apple $x_{ba}=1/5$ 3 carrots → 2 bananas $x_{cb}=2/3$

Start w/ 1 apple, want max # bananas

$a \to b$: $\underline{2}$ bananas x_{ab}

$a \to c \to b$: 1 apple → 6 carrots → $\underline{4}$ bananas $x_{ac}x_{cb}$

Want to maximize $\prod_{e \in P} x_e$ over all directed paths P from $i \to j$.

Take $c_e = -\log(x_e) \Rightarrow -\log\left(\prod_{e \in P} x_e\right) = \sum_{e \in P} c_e$

Since $\log(\cdot)$ is increasing, min cost path $i \to j$ (with cost c_e) gives best exchange rate.

MOORE-BELLMAN-FORD ALGORITHM

<u>Input</u>: digraph $G=(V,E)$, source $s\in V$, edge costs $c:E\to\mathbb{R}$
s.t. G has no negative cycles (Q: how to test this?)

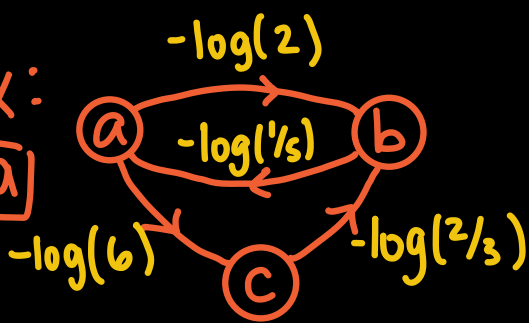
<u>Output</u>: length $\ell(v)$ of shortest s-v path for all $v\in V$

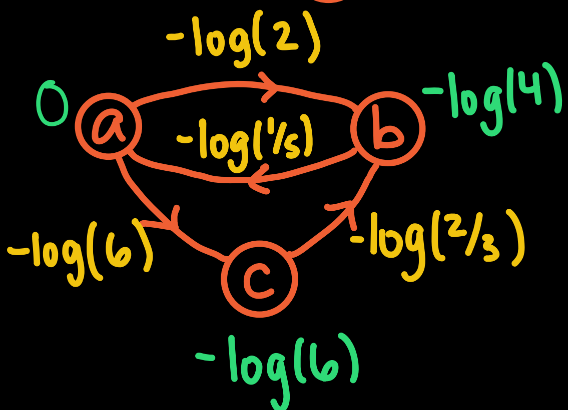
(1) Set $\ell(s)=0$, $\ell(v)=\infty$ for all $v\in V\setminus\{s\}$

(2) For $i=1,\dots,|V|-1$ do:

(3) For each $(v,w)\in E$, $\ell(w):=\min\{\ell(w),\ \ell(v)+c(v,w)\}$

Ex: $\boxed{s=a}$





		$\ell(a)$	$\ell(b)$	$\ell(c)$	
	—	0	∞	∞	
$i=1$	(c,b)	.	.	.	
	(b,a)	.	.	.	
	(a,c)	.	.	$-\log(6)$	
	(a,b)	.	$-\log(2)$	.	
$i=2$	(c,b)	.	$-\log(4)$	.	$\leftarrow -\log(6)-\log(2/3) = -\log(6\cdot\frac{2}{3}) = -\log(4) < -\log(2)$
	(a,c)	.	.	.	
	(b,a)	•	.	.	$\leftarrow -\log(\frac{1}{5})-\log(4) = -\log(4/5) > 0$
	(a,b)	.	.	.	

Thm: Output $\ell(v)$ gives min cost of s-v path in G.

(Proof) Let $d(s,v)$ = min cost of path $s\to v$ in G. (WTS $\ell(v)=d(s,v)$)

At any iteration, $\ell(v)$ = length of some path $s\to v \Rightarrow \ell(v)\geq d(s,v)$

Let $s=v_0\to v_1\to\dots\to v_k=v$ be min cost path $s\to v$

Bellman's Principle: $s=v_0 \to v_1 \to \dots \to v_{j-1} \to v_j$ is min cost $s \to v_j$ path

Claim: After j^{th} iteration of alg, $\ell(v_j) = d(s, v_j)$

$(j=0)$ $\ell(s) = 0$ ✓

$(j>0)$ Assume $\ell(v_j) = d(s, v_j)$ after j^{th} iteration.

In $(j+1)^{st}$ iteration, edge $v_j \to v_{j+1}$ updates

$$\ell(v_{j+1}) = \min\{\ell(v_{j+1}), \underbrace{\ell(v_j) + c(v_j, v_{j+1})}\} = d(s, v_{j+1})$$

$= \text{length of } s \to v_1 \to \dots \to v_{j+1} = d(s, v_{j+1})$

□

Q: How to check for negative cycles?

Application: arbitrage (n products/currencies)

x_{ij} = amt of j that can be attained from 1 unit of i

$c_{ij} = -\log(x_{ij}) \in \mathbb{R}$

Negative cost directed cycle gives opportunity for arbitrage:

$$\sum_{e \in C} c_e < 0 \iff -\sum_{e \in C} \log(x_e) < 0$$

$$\iff \log\left(\prod_{e \in C} x_e\right) > 0$$

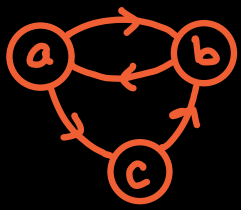
$$\iff \prod_{e \in C} x_e > 1$$

negate and use log rules

Ex (cont'): New exchange rates

1 apple → 2 bananas $x_{ab} = 2$ 1 apple → 6 carrots $x_{ac} = 4$

3 bananas → 1 apple $x_{ba} = 1/3$ 3 carrots → 2 bananas $x_{cb} = 2/3$



1 apple → 6 carrots → 4 bananas → $\frac{4}{3}$ apples !

$$x_{ac}\, x_{cb}\, x_{ba} = 6 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{3}$$

Take $c(e) = -\log(x_e)$

Cost of cycle $a \to c \to b \to a$: $-\log(6) - \log\left(\frac{2}{3}\right) - \log\left(\frac{1}{3}\right)$

$$= -\log\left(\frac{6 \cdot 2}{3 \cdot 3}\right) = -\log\left(\frac{4}{3}\right) < 0$$