Math 409: Discrete Optimization

Today: Directed graphs and shortest paths

DIJKSTRA'S ALGORITHM

Input: $G=(V,E)$ digraph, $s\in V$, costs $c: E\to \mathbb{R}_{\geq 0}$

Output: min cost $\ell(v)$ of s-v path for all $v\in V$

(1) Set $\ell(s)=0$, $R=\{s\}$

For $v\in V\setminus\{s\}$, set $\ell(v)=\begin{cases} c(s,v) & \text{if } (s,v)\in E \\ \infty & \text{if } (s,v)\notin E \end{cases}$

(2) While $R\neq V$ do

(3) select $v\in V\setminus R$ attaining $\min\{\ell(v) : v\in V\setminus R\}$

(4) For $w\in V\setminus R$ with $(v,w)\in E$, do $\ell(w):=\min\{\ell(w), \ell(v)+c(v,w)\}$

(5) Set $R=R\cup\{v\}$

Thm: For every $v\in V$, the output $\ell(v)$ is the length of the shortest s-v path in G.

Take $d(v,w)$ = true shortest length of v-w path in G

(WTS $\ell(v)=d(s,v)$)

Claim: At every step of alg.:

1) For all $u\in R$, $\ell(u)=d(s,u)$

2) For all $u\in V/R$, $\ell(u)$ = length of shortest s-u path in $R\cup\{u\}$

(Induct on $|R|$) $R=\{s\}$ (You check!)

Consider step (3) selecting $v \in V \setminus R$.

WTS (1) and (2) for $R' = R \cup \{v\}$, ℓ = lengths before update

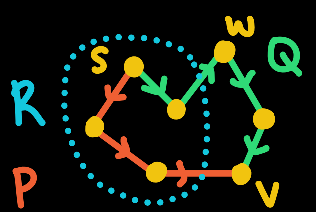
ℓ' = lengths after update

(1) $u \in R' \Rightarrow u \in R$ or $u = v$

($u \in R$) $\ell'(u) = \ell(u) = d(s,u)$ by induction

($u = v$) Let P = shortest s-v path in $R \cup \{v\}$ ← length $\ell(v)$ by (2)

Suppose there's a shorter s-v path Q in G.



Let w = first vertex in $V \setminus R$ in Q

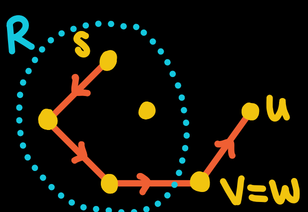
By (2), $\ell(w) \leq$ length of s-w part of Q

$\leq$ full length of Q < length of $P = \ell(v)$

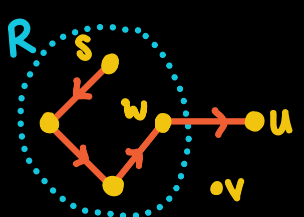
$\ell(w) < \ell(v)$ contradicts our choice of $v \in V \setminus R$ as attaining $\min\{\ell(u) : u \in V \setminus R\}$ $\Rightarrow \ell(v) = d(s,v)$

(2) Take $u \in V / R'$ and Q = Shortest s-u path in $R' \cup \{u\}$

and let $w \in R'$ be the penultimate vertex in Q.



($w = v$) length of Q is $\ell(v) + c((v,u)) \geq \ell'(u)$



($w \neq v$) By (1), $d(s,w) = \ell(w)$ (shortest s-w path $\subseteq R$)

$\Rightarrow$ we can assume $v \notin Q$

$\Rightarrow C(Q) = d(s,w) + c((w,u)) = \ell(u) \leq \ell'(u)$

Thm: Running time of Dijstra's Alg. in $O(n^2)$ $n=|V|$

(Proof) Alg. runs steps (3)-(5) for each $v \in V \setminus \{s\}$ (n times). For each, there are at most n updates $\ell(w) \to \ell'(w)$ for $w \in V \setminus R$, each of which takes $O(1)$.

No known algorithms do better than this!

Dijkstra's alg. needed $c(e) \geq 0 \ \forall e \in E$, run time $O(|V|^2)$

Relaxing to no negative cycles, we only get $O(|V||E|)$

MOORE-BELLMAN-FORD ALGORITHM

Input: digraph $G=(V,E)$, source $s \in V$, edge costs $c: E \to \mathbb{R}$ s.t. G has no negative cycles (Q: how to test this?)

Output: length $\ell(v)$ of shortest s-v path for all $v \in V$

(1) Set $\ell(s)=0$, $\ell(v)=\infty$ for all $v \in V \setminus \{s\}$

(2) For $i=1,..,|V|-1$ do:

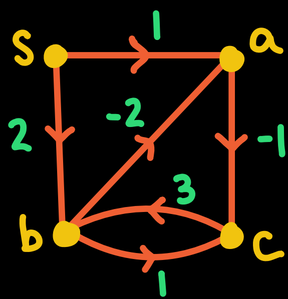
(3) For each $(v,w) \in E$, $\ell(w) := \min\{\ell(w), \ell(v)+c(v,w)\}$

Note: running time is $O(nm)$ where $n=|V|$, $m=|E|$

Run (2) n times, each involves m iterations of (3)

Proof of correctness next time!

Ex:



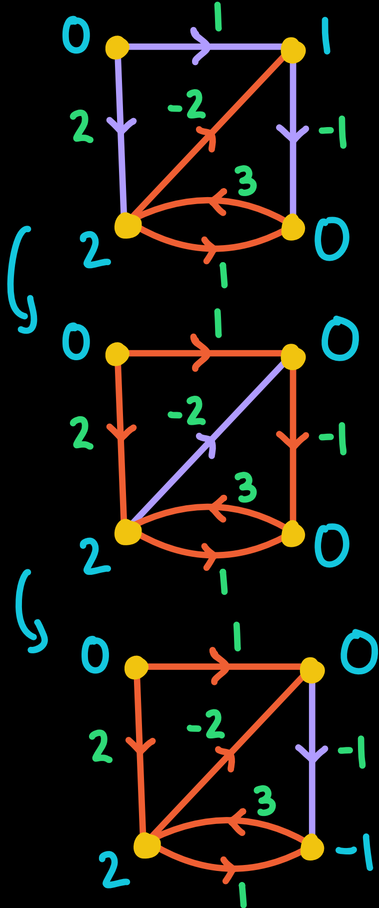
← Note: Running Dijkstra would give $\ell(a)=1$

	(v,w)	$\ell(s)$	$\ell(a)$	$\ell(b)$	$\ell(c)$
	–	0	∞	∞	∞
$i=1$	(s,a)	·	1	·	·
	(s,b)	·	·	2	·
	(a,c)	·	·	·	0
	(b,a)	·	0	·	·
	(b,c)	·	·	·	·
	(c,b)	·	·	·	·
$i=2$	(s,a)	·	·	·	·
	(s,b)	·	·	·	·
	(a,c)	·	·	·	-1
	(b,a)	·	·	·	·
	(b,c)	·	·	·	·
	(c,b)	·	·	·	·

· = no change

$\min\{0, 3+1\} = 0$

$\min\{2, 0+3\} = 2$



Should do $i=3$, but nothing changes

FINAL: 0 0 2 -1