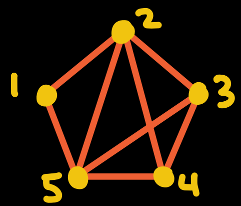
Math 409: Discrete Optimization

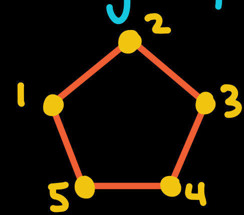
Today: Traveling Salesman Problem (TSP)

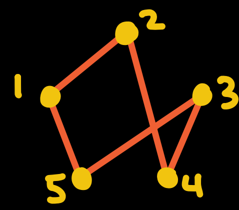
Recall: A <u>Hamiltonian circuit</u> (or <u>tour</u>) of a graph G is a spanning cycle subgraph of G.

Ex: G



Two Hamiltonian circuits of G:





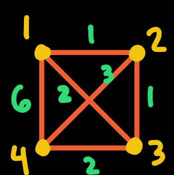
<u>TRAVELING SALESMAN PROBLEM (TSP)</u>

See TSP applications online!

<u>Input</u>: positive integer n and edge costs $c_{ij} \in \mathbb{R}_{\geq 0}$ for every $1 \leq i < j \leq n$

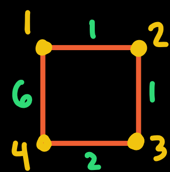
<u>Goal</u>: Find a Hamiltonian circuit C of K_n minimizing the total cost $\sum_{e \in C} c_e$

Ex: $(n=4)$

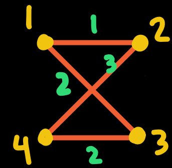


$c(12)=1$ $c(23)=1$
$c(13)=2$ $c(24)=3$
$c(14)=6$ $c(34)=2$

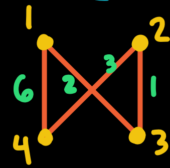
Total cost



$1+1+2+6 = 10$



$\boxed{1+2+3+2 = 8}$



$1+2+3+6 = 12$

One way to solve: list all tours of K_n and compare

How many tours are there?

Starting from $v_0 = 1$, there are $(n-1)$ choices for v_1
$(n-2)$ choices for v_2

Each Hamiltonian circuit has two such representations $(v_0, v_1, \dots, v_{n-1})$ and $(v_0, v_{n-1}, \dots, v_2, v_1)$

$\vdots$
$(n-k)$ choices for v_k
$\vdots$
1 choice for v_{n-1}

$\Rightarrow$ # Ham. circuits of $K_n = \frac{1}{2}(n-1)!$

Grows really fast!
Stirlings formula
$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

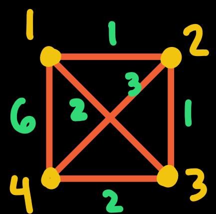
A (slightly) better way (Held-Karp Algorithm)

Idea: iteratively compute best subtours

Define the cost of a path P as $\sum_{e \in E(P)} c_e$

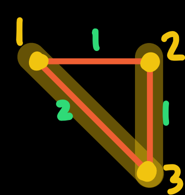
For any subset $S \subseteq V$ of vertices and $i, j \in S$, define

$C(S, i, j) =$ min. cost of a path from i to j in S visiting each vertex of S

Ex: 

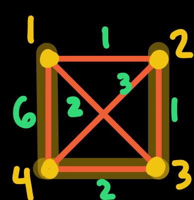
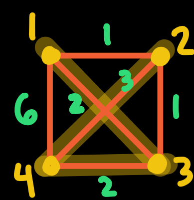
$S = \{1,2,3\}$ $i=1$, $j=2$

$C(S,i,j) = 3$



$S = \{1,2,3,4\}$ $i=1$, $j=2$

$C(S,i,j) = \min\{9, 7\} = 7$

Claim: Min cost tour $= \min_{j \neq 1} \left(C(V, 1, j) + c_{1j} \right)$

(Proof sketch) ($\leq$) Add $\{1,j\}$ to path attaining $C(V,1,j)$

($\geq$) Any tour contains an edge $\{1,j\}$ for some $j \neq 1$.

HELD-KARP ALGORITHM

Input: $n \geq 3$, costs $c_{ij} \geq 0$ for $1 \leq i < j \leq n$

Output: min cost of a tour on K_n

(1) $C(\{i,j\}, i, j) := c_{ij}$ for all $i \neq j$

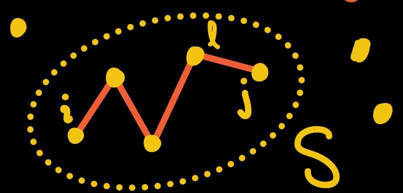
(2) For $k = 3$ to n do:

(3) For all sets $S \subseteq V$ with $|S| = k$, do:

$$C(S,i,j) := \min_{\ell \in S \setminus \{i,j\}} \left(C(S \setminus \{j\}, i, \ell) + c_{\ell j} \right)$$

(4) Output $\min_{j \neq 1} C(V,1,j) + c_{1j}$

Idea for (3): Every path $i \to j$ in S has final edge $\{\ell, j\}$ for some $\ell \in S \setminus \{i,j\}$



By induction on $|S|$, $C(S \setminus \{j\}, i, \ell)$ is min cost path $i \to \ell$ in $S \setminus \{j\}$.

$\Rightarrow$ $C(S \setminus \{j\}, i, \ell) + c_{\ell j}$ is min cost path $i \to j$ in S with final edge $\{\ell, j\}$.

$\Rightarrow$ min over all possible $\ell \in S \setminus \{i,j\}$ gives $C(S,i,j)$.

By Claim above, output is min cost TSP tour

What is the running time?

Claim: Held-Karp runs in time $O(n^3 2^n)$.

$\left.\begin{array}{l}\#\text{subsets } S \subseteq \{1,\dots,n\} = 2^n \\ \#\text{pairs } (i,j) = n(n-1)\end{array}\right\} \Rightarrow$ need to compute $O(n^2 2^n)$ values $C(S,i,j)$

For each, we compute $\min_{\ell \in S \setminus \{i,j\}} \left(C(S \setminus \{j\}, i, \ell) + c_{\ell j} \right)$.

Need to compute $\leq n$ terms $C(S \setminus \{j\}, i, \ell) + c_{\ell j}$ and take minimum $\rightsquigarrow O(n)$ at each step (3)

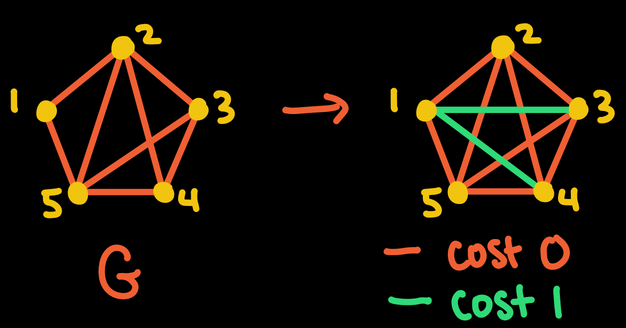
$\Rightarrow$ Total run time in $O(n^3 2^n)$

It is unknown if TSP can solved in time $O(1.99^n)$

Better bounds are known in special cases.

Finding a Hamiltonian circuit in $G=(V,E)$

Idea: Define $c_e = 0$ for $e \in E$ and $c_e = 1$ for $e \subseteq V, e \notin E$



Original graph G has a Hamiltonian circuit $\Longleftrightarrow 0 =$ min cost of a TSP tour on K_V.