Math 409: Discrete Optimization

Today: Intro to graphs

Many definitions:

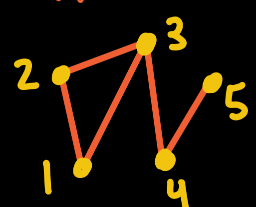
undirected graph (or just graph)

$G=(V,E)$ where elt. of E have the form $\{i,j\}$ with $i,j \in V$

$V=V(G)=$ vertices of G

$E=E(G)=$ edges of G

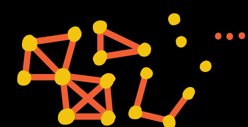
Ex: $V=\{1,2,3,4,5\}$



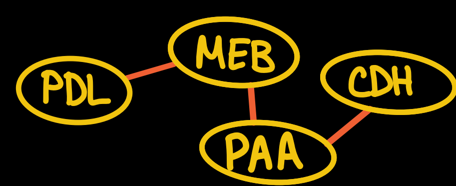
$E=\{\{1,2\},\{1,3\},\{2,3\},\{3,4\},\{4,5\}\}$

abbrev $E=\{12,13,23,34,45\}$

Ex: $V=$ students in this class



$E=$ pairs of students taking another course together

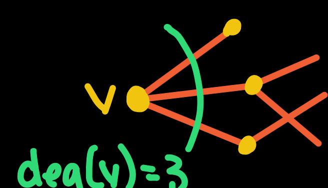


Ex: $V=$ buildings on UW campus

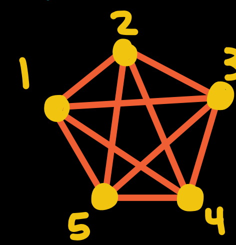
$E=$ pairs you can travel between in ≤ 10 min

Degree of a vertex $v \in V$:

$\deg(v)=\#\{e \in E \text{ s.t. } v \in e\}$



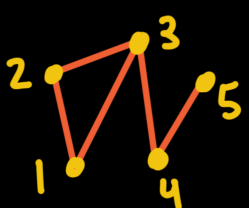
The <u>complete graph</u> on V: graph with edges $\{i,j\}$ for all distinct $i,j \in V$. Denoted K_V or K_n for $V=\{1,\dots,n\}$



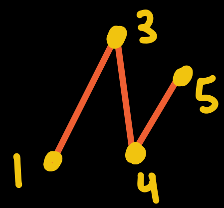
K_5

A <u>subgraph</u> of $G=(V(G),E(G))$ is a graph $H=(V(H),E(H))$ where $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$.

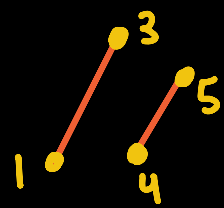
H is an <u>induced subgraph</u> if $E(H)=\{e\in E(G) \text{ s.t. } e\subseteq V(H)\}$ (determined by $V(H)$ and G)



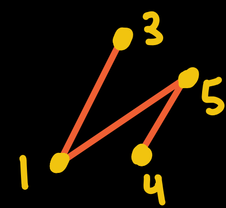
G



induced subgraph

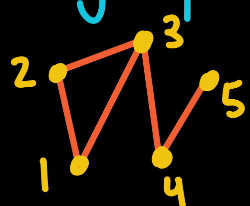


(non-induced) subgraph

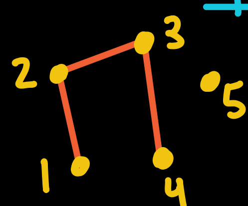


not a subgraph

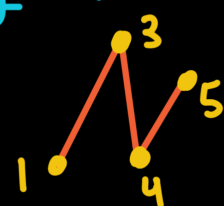
A subgraph H of G is <u>spanning</u> if $V(H)=V(G)$



G



spanning



not spanning

Subgraphs from <u>deletions</u>:

For $G=(V,E)$, $V'\subseteq V$, $E'\subseteq E$:

$G\setminus E' = (V, E\setminus E')$

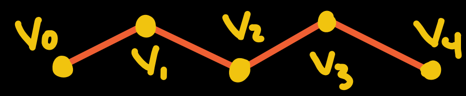
$G\setminus V' = (V\setminus V', \{e\in E : e\cap V' = \emptyset\})$ ← induced subgraph on $V\setminus V'$

Special (sub) graphs

A <u>path</u> is a graph $P=(V,E)$ where $V=\{v_0,\dots,v_k\}$
$E=\{\{v_{i-1},v_i\}: i=1,\dots,k\}$ with $v_0,\dots,v_k$ all distinct.

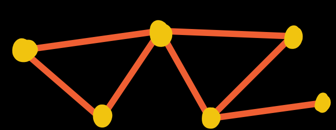
A <u>(v_0,v_k)-path</u> is a path with end points v_0, v_k

<u>length</u> $(P)=\#E=k$

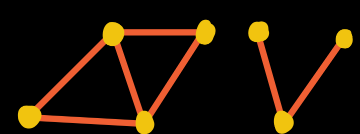


A graph is <u>connected</u> if it contains (i.e. has as a subgraph) a path between any two of its vertices.

A <u>connected component</u> of G is a subset $V' \subseteq V$ s.t. there is a path between any pair of vertices in V' but no edge between $u \in V'$ and $v \notin V'$



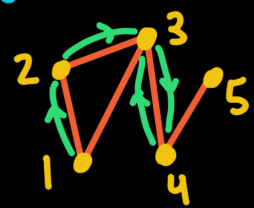
connected



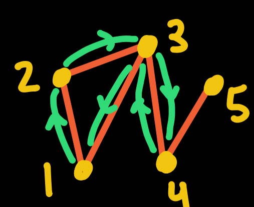
disconnected (2 connected components)

A <u>walk</u> in $G=(V,E)$ is a sequence $(v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k)$ with $v_i \in V$, $e_i=\{v_{i-1}, v_i\} \in E$.
(determined by $(v_0, v_1, \dots, v_k)$)   <u>length</u> = # edges = k

Ex: $(1, \{1,2\}, 2, \{2,3\}, 3, \{3,4\}, 4, \{3,4\}, 3)$
↳ just vertices $(1,2,3,4,3)$

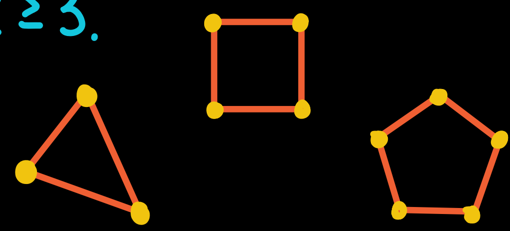


If $v_0=v_k$, this is a <u>closed walk</u>.



e.g. $(1,2,3,4,3,1)$

A <u>cycle</u> (or <u>k-cycle</u>) is a graph $G=(V,E)$ with $V=\{v_0,\dots,v_{k-1}\}$ and $E=\{\{v_0,v_1\},\dots,\{v_{k-2},v_{k-1}\},\{v_{k-1},v_0\}\}$ where $v_0,\dots,v_{k-1}$ are distinct and $k\geq 3$.

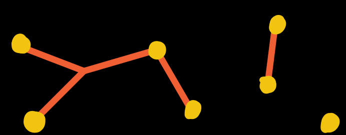


C_n = cycle on vert. $\{1,\dots,n\}$ w/ edges $\{i,i+1\}$, $\{1,n\}$

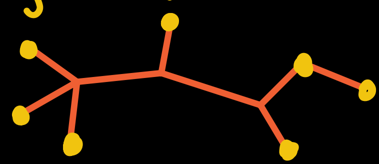
G is <u>acyclic</u> if no subgraph of G is a cycle

<u>forest</u> = acyclic graph   <u>tree</u> = connected forest

↑ also = union of trees on disjoint vertex sets

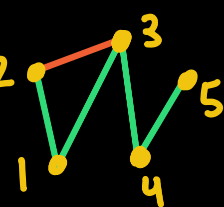


forest

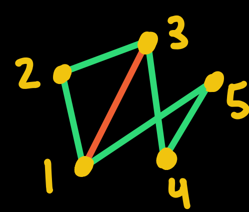


tree

A <u>spanning tree</u> of $G=(V,E)$ is a spanning, connected, acyclic subgraph

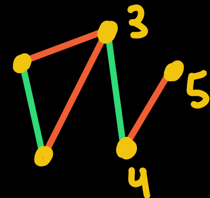


A <u>Hamiltonian circuit</u> (or <u>tour</u>) of G is a spanning cycle subgraph of G.

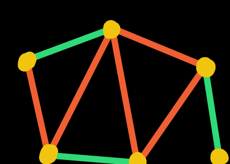


A <u>matching</u> of $G=(V,E)$ is a set $M\subseteq E$ in which every vertex has $\deg \leq 1$.

M



If every vertex has $\deg = 1$, M is a <u>perfect matching</u>



M