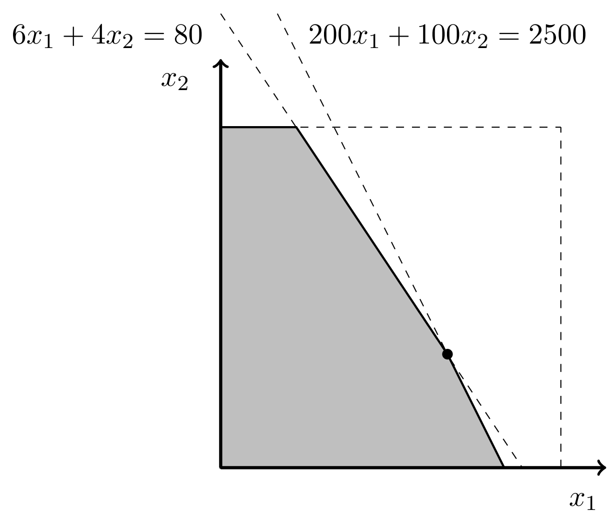
Math 407 – Sensitivity Analysis

Wednesday, November 29

Changes in the right-hand side: $\mathbf{b} \to \tilde{\mathbf{b}}$

Our small craft store is still making scarves and hats and trying to maximize their profits. Every week, they have 80 hours of manual labor and 2500g of wool available.

	work (hours)	wool (g)	demand (#)	profit ($)
scarf	6	200	15	10
hat	4	100	15	6



As a linear program:

$$\text{(OP)} \quad \max \quad 10x_1 + 6x_2 \quad \text{s.t.} \quad \begin{array}{l} 0 \le x_1 \le 15 \\ 0 \le x_2 \le 15 \\ 6x_1 + 4x_2 \le 80 \\ 200x_1 + 100x_2 \le 2500 \end{array}$$

where

$x_1 =$ (average) number of scarves made per week

$x_2 =$ (average) number of hats made per week

optimal solution: $(x_1, x_2) = (10, 5)$

optimal value: 130

Suppose that the amount of work hours available each week changes $80 \to (80 + t)$.

Q1: For what values of t is it feasible to use up all available work hours and wool?

Q2: For t in this range, how do the optimal solution and optimal value change with t?

Q3: How much additional profit can be gained from having 8 additional hours of work?

Q4: What is the dual LP of the original linear program?

Q5: What is the optimal solution of the dual?

Q6: How does changing the available work hours $80 \to (80 + t)$ affect the dual LP?

Q7: How would changing the available wool from 2500g to $(2500 + t)$g affect the dual LP? What about the optimal value?

Q8: How much additional profit can be gained from an additional 100g of wool?

Solutions

A1: Using up all available work hours and wool means that

$$6x_1^* + 4x_2^* = 80 + t$$
$$200x_1^* + 100x_2^* = 2500$$

meaning that

$$\begin{pmatrix} 6 & 4 \\ 200 & 100 \end{pmatrix} \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 80+t \\ 2500 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 200 & 100 \end{pmatrix}^{-1} \begin{pmatrix} 80+t \\ 2500 \end{pmatrix}$$
$$= \begin{pmatrix} -1/2 & 1/50 \\ 1 & -3/100 \end{pmatrix} \begin{pmatrix} 80+t \\ 2500 \end{pmatrix} = \begin{pmatrix} 10-t/2 \\ 5+t \end{pmatrix}.$$

This is feasible if and only this point satisfies the other four inequalities: $0 \le x_1 \le 15$, $0 \le x_2 \le 15$. Plugging in $x_1 = 10 - t/2$ and $x_2 = 5 + t$ gives the conditions:

$$\begin{aligned} 0 \le 10 - t/2 \quad &\Leftrightarrow \quad t \le 20 \\ 10 - t/2 \le 15 \quad &\Leftrightarrow \quad -10 \le t \\ 0 \le 5 + t \quad &\Leftrightarrow \quad -5 \le t \\ 5 + t \le 15 \quad &\Leftrightarrow \quad t \le 10 \end{aligned}$$

Therefore this point is feasible for $-5 \le t \le 10$.

A2: Since the cost vector $(10, 6)$ is a positive combination of $(6, 4)$ and $(200, 100)$, namely

$$\begin{pmatrix} 10 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \frac{1}{50} \begin{pmatrix} 200 \\ 100 \end{pmatrix},$$

the point $(x_1, x_2) = (10 - t/2, 5 + t)$ found above is optimal if and only if it is feasible!
Recall: Given a linear program $\max\{\mathbf{c}^T\mathbf{x} \text{ s.t. } A\mathbf{x} \le \mathbf{b}\}$ and a feasible point $\mathbf{x}^*$ of (P),

$$\mathbf{x}^* \text{ is optimal } \Leftrightarrow \mathbf{c} = \sum_{i \in I} y_i \mathbf{a}_i \text{ for some } y_i \ge 0$$

where $I = \{i : \mathbf{a}_i^T\mathbf{x}^* = b_i\}$.
(Why? Duality and complimentary slackness!: $\mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x}^* = \sum_{i=1}^m y_i(b_i - \mathbf{a}_i^T\mathbf{x}^*)$)

Therefore $-5 \le t \le 10$,

$$\begin{aligned} \text{Optimal Solution: } & (10 - t/2, 5 + t) \\ \text{Optimal Value: } & 10(10 - t/2) + 6(5 + t) = 130 + t \end{aligned}$$

A3: Adding 8 additional hours of work gives an additional \$8 of profit.
(To maximize profits, they should make $6 = 10 - 8/2$ scarves and $13 = 5 + 8$ hats per week for a total profit of $\$138 = 10 \cdot 6 + 6 \cdot 13$.)

A4: The dual LP of the original linear program is:

$$
\text{(OD)} \quad \max \quad 15y_2 + 15y_4 + 80y_5 + 2500y_6 \quad \text{s.t.} \quad \begin{array}{l} y_1, y_2, y_3, y_4, y_5, y_6 \geq 0 \\ -y_1 + y_2 + 6y_5 + 200y_6 = 10 \\ -y_3 + y_4 + 4y_5 + 100y_6 = 6 \end{array}
$$

A5: In the optimal solution for the primal, only the 5th and 6th inequalities hold with equality at $(x_1^*, x_2^*) = (10, 5)$. By complimentary slackness, the optimal solution for the dual has $y_1 = y_2 = y_3 = y_4 = 0$. Therefore the optimal solution for the dual is given by

$$
\begin{aligned}
6y_5 + 200y_6 = 10 \qquad &\Rightarrow \qquad y_5 = 1 \\
4y_5 + 100y_6 = 6 \qquad &\Rightarrow \qquad y_6 = 1/50
\end{aligned}
$$

Therefore the optimal solution for the dual is:

$$
\begin{pmatrix} y_1^* \\ y_2^* \\ y_3^* \\ y_4^* \\ y_5^* \\ y_6^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1/50 \end{pmatrix}
$$

A6: Changing $80 \to (80 + t)$ hours changes the *objective function* of the dual LP. As long as $\mathbf{y}^* = (0, 0, 0, 0, 1, 1/50)^T$ remains the optimal solution for the dual, the optimal value of $15y_2^* + 15y_4^* + (80 + t)y_5^* + 2500y_6^*$ will be

$$
(80 + t)(1) + 2500(1/50) = 130 + t.
$$

Note: If $\mathbf{y}^*$ is the unique optimal solution for the dual LP, then it will remain the optimal solution for small changes in $\mathbf{b}$. We can use $\mathbf{y}^*$ to determine the change in optimal value:

$$
\text{new optimal value} = \tilde{\mathbf{b}}^T \mathbf{y}^* = (\mathbf{b} + \Delta\mathbf{b})^T \mathbf{y}^* = \mathbf{b}^T \mathbf{y}^* + \Delta\mathbf{b}^T \mathbf{y}^*
$$

In the example above, we see that increasing b_5 by t increases the optimal value by t because $y_5^* = 1$:

$$
\text{new optimal value} = \begin{pmatrix} 0 \\ 15 \\ 0 \\ 15 \\ 80 + t \\ 2500 \end{pmatrix}^T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1/50 \end{pmatrix} = 130 + t.
$$

A7: Similarly, As long as $\mathbf{y}^* = (0, 0, 0, 0, 1, 1/50)^T$ remains the optimal solution for the dual LP, changing the grams of available wool $2500 \to (2500 + t)$ will give a maximum profit of

$$
15y_2^* + 15y_4^* + 80y_5^* + (2500 + t)y_6^* = 80(1) + (2500 + t)(1/50) = 130 + t/50
$$

In particular, the coefficient, $1/50$, of t comes from y_6^*.

A8: With an additional 100g of wool we can check that it is still feasible ($\Rightarrow$ optimal) to use up all available work hours and wool:

$$\begin{pmatrix} 6 & 4 \\ 200 & 100 \end{pmatrix} \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 80 \\ 2600 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 200 & 100 \end{pmatrix}^{-1} \begin{pmatrix} 80 \\ 2600 \end{pmatrix}$$
$$= \begin{pmatrix} -1/2 & 1/50 \\ 1 & -3/100 \end{pmatrix} \begin{pmatrix} 80 \\ 2600 \end{pmatrix} = \begin{pmatrix} 12 \\ 2 \end{pmatrix}.$$

This is still feasible since both coordinates are between 0 and 15. Therefore the maximum profit increases by $\$2 = 100/50$.