Math 407B/D – Final Exam
Autumn 2022

Time: 110 mins.

1. This exam is closed book and closed notes. You may not use any calculators, technological devices, or outside materials besides a writing utensil.

2. Stop writing when time is up. If you do not, points my be deducted from your final score.

3. Answer all questions in the spaces provided. If you run out of room for an answer, continue on the back of the page. There are also two blank pages for scratch work at the end of the exam. Do not tear these off.

4. Unless stated otherwise, justify your answers to receive full credit. Your answers do not have to be in complete sentences, but they do need to be understandable.

Name: _____

Student ID #: _____

Question	Points	Score
1	25	
2	20	
3	25	
4	12	
5	18	
Total:	100	

Possibly useful: The inverse of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{(ad - bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

1. (25 points) Parts (a) - (f) refer to the following linear program:

$$\max \quad 2x_1 + 3x_2 \quad \text{such that} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad 1 \leq x_1 + x_2 \leq 3, \quad x_2 \leq 2$$

(a) Plot the feasible region of this linear program and solve it graphically.
    (An appropriately drawn and labeled line in your picture can serve as justification.)

x_2

optimal solution =

x_1
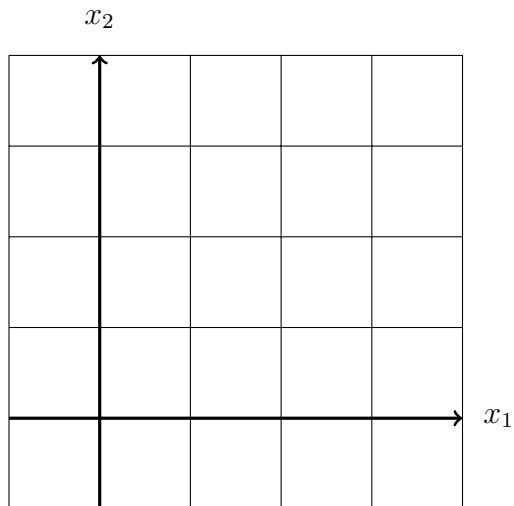
optimal value =

(b) Give a matrix A and vectors $\mathbf{b}$, $\mathbf{c}$ so that $\max \ \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$ is a reformulation of this linear program.

(c) What is an optimal solution to your LP from (b) and corresponding basis?

1. (continued) Parts (a) - (f) refer to the following linear program:

$$\max \quad 2x_1 + 3x_2 \quad \text{such that} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad 1 \leq x_1 + x_2 \leq 3, \quad x_2 \leq 2$$

(d) Formulate the dual linear program from the primal in inequality form (as above).

(e) Use complimentary slackness to find an optimal solution of the dual.

(f) For what values of $t \in \mathbb{R}$ is the maximum of $2x_1 + tx_2$ over this feasible region attained at the optimal solution in your answer from (a)?

2. (20 points) In each part (a) - (d), you are given a simplex tableau of a linear program in equational form. For each, you should either conclude that the linear program is 1) infeasible, 2) unbounded, or 3) both feasible and bounded, in which case you should **find the optimal solution**. If needed, perform a pivot step in either the usual simplex method or dual simplex method.

(a)
$$x_4 = 2 + x_1 - x_2$$
$$x_5 = 1 + x_1 \qquad + x_3$$
$$x_6 = 2 \qquad + x_2 - x_3$$
$$z = 5 + x_1 - 2x_2 - 4x_3$$

(b)
$$x_4 = 2 - x_1 - x_2$$
$$x_5 = 1 + x_1 \qquad + x_3$$
$$x_6 = 2 \qquad + x_2 - x_3$$
$$z = 5 + x_1 - 2x_2 - 4x_3$$

2. (continued) In each part (a) - (d), you are given a simplex tableau of a linear program in equational form. For each, you should either conclude that the linear program is 1) infeasible, 2) unbounded, or 3) both feasible and bounded, in which case you should **find the optimal solution**. If needed, perform a pivot step in either the usual simplex method or dual simplex method.

(c)
$$x_4 = 2 - x_1 - x_2$$
$$x_5 = 1 + x_1 \qquad + x_3$$
$$x_6 = 2 \qquad + x_2 - x_3$$
$$z = 5 - x_1 - 2x_2 - 4x_3$$

(d)
$$x_4 = \quad 2 - x_1 - x_2$$
$$x_5 = -1 + x_1 \qquad + x_3$$
$$x_6 = \quad 2 \qquad + x_2 - x_3$$
$$z = \quad 5 - x_1 - 2x_2 - 4x_3$$

3. (25 points) A bakery sells baguettes, crackers, and loaves of bread.

- A batch of baguettes requires 1 hour of work, 1kg of flour, and yields \$7 profit.
- A batch of crackers requires 1 hour of work, 2kg of flour, and yields \$9 profit.
- A batch of loaves of bread requires 1 hour of work, 3kg flour, and yields \$10 profit.

The bakery has 40 hours of labor and 100kg of flour available each day.
Use this data to answer (a) - (e) below.

(a) Using x_1, x_2, x_3 (respectively) to be the number of batches of baguettes, crackers, and loaves of bread made daily and adding slack variables as needed, write a linear program in *equational form* to find the maximum profit of the bakery per day.

(b) To maximize their profit every day, the bakery makes no baguettes, 20 batches of crackers, and 20 batches of loaves of bread. Give the basis and simplex tableau corresponding to this solution.

(c) The bakery is considering increasing the number of hours worked each day. What is the maximum number of hours they can add per day so that they still maximize profits by making no baguettes and using up all of their available resources?

(d) If they add fewer hours than your answer from (c), how much additional profit can they make for every extra hour added?

(e) Starting from the original data, what is the minimum *profit per batch of baguettes* for which there is a solution maximizing profits that involves making (a positive number of) baguettes?

4. (12 points) Consider the linear program

$$\max \ \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad x_2 \le 2x_1, \ \ x_1 \ge -1, \ \text{and} \ x_2 \le 2,$$

some of whose feasible region is shown below.
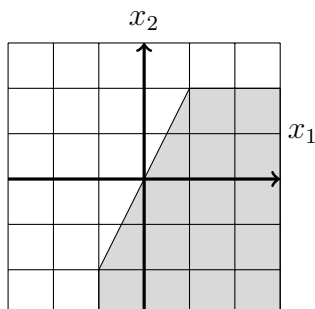
In each part below, **circle all** choices of $\mathbf{c}^T$ that make the linear program or its dual have the requested property.

Do not justify your answers.

(a) The linear program attains its optimal value at $(x_1, x_2) = (1, 2)$.

$$\mathbf{c}^T = \big(-2, 1\big) \qquad \mathbf{c}^T = \big(-1, 1\big) \qquad \mathbf{c}^T = \big(0, 1\big) \qquad \mathbf{c}^T = \big(1, 1\big) \qquad \mathbf{c}^T = \big(1, 0\big)$$

(b) The linear program has more than one optimal solution.

$$\mathbf{c}^T = \big(-2, 1\big) \qquad \mathbf{c}^T = \big(-1, 1\big) \qquad \mathbf{c}^T = \big(0, 1\big) \qquad \mathbf{c}^T = \big(1, 1\big) \qquad \mathbf{c}^T = \big(1, 0\big)$$

(c) The dual linear program is infeasible.

$$\mathbf{c}^T = \big(-2, 1\big) \qquad \mathbf{c}^T = \big(-1, 1\big) \qquad \mathbf{c}^T = \big(0, 1\big) \qquad \mathbf{c}^T = \big(1, 1\big) \qquad \mathbf{c}^T = \big(1, 0\big)$$

(d) The dual linear program is unbounded (from below).

$$\mathbf{c}^T = \big(-2, 1\big) \qquad \mathbf{c}^T = \big(-1, 1\big) \qquad \mathbf{c}^T = \big(0, 1\big) \qquad \mathbf{c}^T = \big(1, 1\big) \qquad \mathbf{c}^T = \big(1, 0\big)$$

5. (18 points) In each part below, circle True or False. Do not justify your answers.

(a) A linear program is feasible if and only if its auxiliary linear program is bounded.

True   False

(b) Every polyhedron has a vertex.

True   False

(c) If $\mathbf{v}$ is a point in a polyhedron P and $\mathbf{v}$ is not a vertex of P, then there exists a nonzero vector $\mathbf{w} \in \mathbb{R}^n$ and $\varepsilon > 0$ so that both $\mathbf{v} + \varepsilon\mathbf{w}$ and $\mathbf{v} - \varepsilon\mathbf{w}$ belong to P.

True   False

(d) If $\mathbf{c}^T\mathbf{x}$ and $\tilde{\mathbf{c}}^T\mathbf{x}$ both achieve their maximum over a polyhedron P at $\mathbf{x}^*$, then for any $\lambda \geq 0$, $\mu \geq 0$, the maximum of $(\lambda\mathbf{c} + \mu\tilde{\mathbf{c}})^T\mathbf{x}$ over P is also achieved at $\mathbf{x}^*$.

True   False

(e) A subset $B \subseteq \{1, \ldots, n\}$ is a feasible basis of $\max \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$ with $A \in \mathbb{R}^{m \times n}$ if and only if the matrix A_B is invertible.

True   False

(f) Using Bland's pivot rule (choose the smallest index whenever there is a choice), the simplex method can return to previously visited basic feasible solutions.

True   False

(g) If $\mathbf{c}^T$ can be written as a nonnegative linear combination of rows of A, then $\mathbf{c}^T\mathbf{x}$ bounded from above on $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$.

True   False

(h) A linear program attains an optimal value if and only if its dual linear program attains the same optimal value.

True   False

(i) If a linear program is infeasible, then its dual linear program is unbounded.

True   False