Math 407B/D – Final Exam

Autumn 2022

Time: 110 mins.

1. This exam is closed book and closed notes. You may not use any calculators, technological devices, or outside materials besides a writing utensil.
2. Stop writing when time is up. If you do not, points my be deducted from your final score.
3. Answer all questions in the spaces provided. If you run out of room for an answer, continue on the back of the page. There are also two blank pages for scratch work at the end of the exam. Do not tear these off.
4. Unless stated otherwise, justify your answers to receive full credit. Your answers do not have to be in complete sentences, but they do need to be understandable.

Name: ______________________________

Student ID #: ___________________________

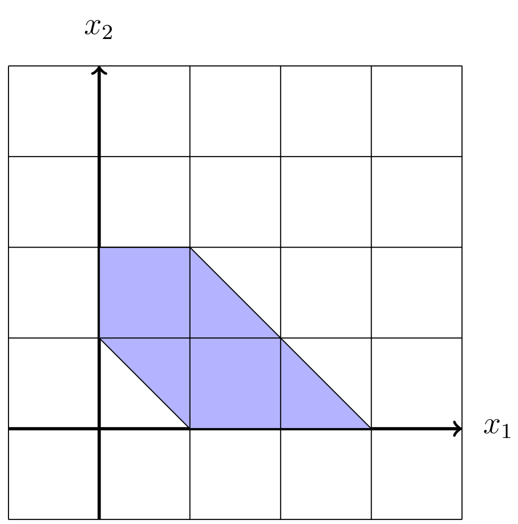
Question	Points	Score
1	25	
2	20	
3	25	
4	12	
5	18	
Total:	100	

Possibly useful: The inverse of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

1. (25 points) Parts (a) - (f) refer to the following linear program:

$$\max \quad 2x_1 + 3x_2 \quad \text{such that} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad 1 \leq x_1 + x_2 \leq 3, \quad x_2 \leq 2$$

(a) Plot the feasible region of this linear program and solve it graphically.
(An appropriately drawn and labeled line in your picture can serve as justification.)



optimal solution = (1,2)

optimal value = 8

(b) Give a matrix A and vectors $\mathbf{b}$, $\mathbf{c}$ so that max $\mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$ is a reformulation of this linear program.

$$A = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(c) What is an optimal solution to your LP from (b) and corresponding basis?

Optimal solution: $(1, 2, 2, 0, 0)$ Basis: $\{1, 2, 3\}$

1. (continued) Parts (a) - (f) refer to the following linear program:

$$\max \quad 2x_1 + 3x_2 \quad \text{such that} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad 1 \leq x_1 + x_2 \leq 3, \quad x_2 \leq 2$$

(d) Formulate the dual linear program from the primal in inequality form (as above).

$$\min -y_3 + 3y_4 + 2y_5 \text{ s.t. } -y_1 - y_3 + y_4 = 2, -y_2 - y_3 + y_4 + y_5 = 3, \mathbf{y} \geq 0$$

$$\min -y_1 + 3y_2 + 2y_3, \text{ s.t. } -y_1 + y_2 \geq 2, -y_1 + y_2 + y_3 \geq 3, \mathbf{y} \geq 0$$

(e) Use complimentary slackness to find an optimal solution of the dual.

First formulation: $\mathbf{y}^* = (0, 0, 0, 2, 1)$
tight inequalities are $x_1 + x_2 \leq 3$ and $x_2 \leq 2 \Rightarrow y_1^* = y_2^* = y_3^* = 0$ and
$(2, 3) = y_4^*(1, 1) + y_5^*(0, 1) = (y_4^*, y_4^* + y_5^*) \Rightarrow y_4^* = 2, y_5^* = 1$

Second formulation: $\mathbf{y}^* = (0, 2, 1)$,

(f) For what values of $t \in \mathbb{R}$ does maximizing $2x_1 + tx_2$ over this feasible region have the same optimal solution as your answer from (a)?

Answer: $t \geq 2$

Possible justification 1: graphical

Possible justification 2: $\exists y_4, y_5 \geq 0$ s.t.

$$\begin{pmatrix} 2 \\ t \end{pmatrix} = \begin{pmatrix} y_4 \\ y_4 + y_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_4 \\ y_5 \end{pmatrix} \Leftrightarrow \begin{pmatrix} y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} 2 \\ t-2 \end{pmatrix} \geq 0$$

Possible justification 3: For $\mathbf{c}^T = (2, t, 0, 0, 0)$ and $B = \{1, 2, 3\}$,

$$\mathbf{c}_N^T - \mathbf{c}_B^T A_B^{-1} A_N = \begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} 2 & t & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & t & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & t-2 \end{pmatrix}$$

2. (20 points) In each part (a) - (d), you are given a simplex tableau of a linear program in equational form. For each, you should either conclude that the linear program is 1) infeasible, 2) unbounded, or 3) both feasible and bounded, in which case you should **find the optimal solution**. If needed, perform a pivot step in either the usual simplex method or dual simplex method.

(a)
$$\begin{aligned} x_4 &= 2 + x_1 - x_2 \\ x_5 &= 1 + x_1 \phantom{{}-x_2} + x_3 \\ x_6 &= 2 \phantom{{}+x_1} + x_2 - x_3 \\ z &= 5 + x_1 - 2x_2 - 4x_3 \end{aligned}$$

Basis is feasible, but not dual feasible $\rightarrow$ pivot.

x_1 enters, no upper bound on x_1 $\rightarrow$ LP is unbounded.

(b)
$$\begin{aligned} x_4 &= 2 - x_1 - x_2 \\ x_5 &= 1 + x_1 \phantom{{}-x_2} + x_3 \\ x_6 &= 2 \phantom{{}+x_1} + x_2 - x_3 \\ z &= 5 + x_1 - 2x_2 - 4x_3 \end{aligned}$$

Basis is feasible, but not dual feasible $\rightarrow$ pivot.

x_1 enters, x_4 leaves $\rightarrow$ new basis: $\{1, 5, 6\}$

$$\begin{aligned} x_1 &= 2 - x_2 - x_4 \\ x_5 &= 3 - x_2 + x_3 - x_4 \\ x_6 &= 2 + x_2 - x_3 \\ z &= 7 - 3x_2 - 4x_3 - x_4 \end{aligned}$$

feasible, bounded, optimal solution: $\mathbf{x}^* = (2, 0, 0, 0, 3, 2)$

2. (continued) In each part (a) - (d), you are given a simplex tableau of a linear program in equational form. For each, you should either conclude that the linear program is 1) infeasible, 2) unbounded, or 3) both feasible and bounded, in which case you should **find the optimal solution**. If needed, perform a pivot step in either the usual simplex method or dual simplex method.

(c)
$$\begin{aligned} x_4 &= 2 - x_1 - x_2 \\ x_5 &= 1 + x_1 \phantom{{}-x_2} + x_3 \\ x_6 &= 2 \phantom{{}-x_1} + x_2 - x_3 \\ z &= 5 - x_1 - 2x_2 - 4x_3 \end{aligned}$$

feasible, bounded, optimal solution: $\mathbf{x}^* = (0, 0, 0, 2, 1, 2)$

(d)
$$\begin{aligned} x_4 &= 2 - x_1 - x_2 \\ x_5 &= -1 + x_1 \phantom{{}-x_2} + x_3 \\ x_6 &= 2 \phantom{{}-x_1} + x_2 - x_3 \\ z &= 5 - x_1 - 2x_2 - 4x_3 \end{aligned}$$

Basis is dual feasible, but not feasible $\rightarrow$ pivot in dual simplex method

x_5 leaves, x_1 enters $\rightarrow$ new basis: $\{1, 4, 6\}$

$$\begin{aligned} x_1 &= 1 - x_3 + x_5 \\ x_4 &= 1 - x_2 + x_3 - x_5 \\ x_6 &= 2 + x_2 - x_3 \\ z &= 4 - 2x_2 - 3x_3 - x_5 \end{aligned}$$

feasible, bounded, optimal solution: $\mathbf{x}^* = (1, 0, 0, 1, 0, 2)$

3. (25 points) A bakery sells baguettes, crackers, and loaves of bread.

   - A batch of baguettes requires 1 hour of work, 1kg of flour, and yields \$7 profit.
   - A batch of crackers requires 1 hour of work, 2kg of flour, and yields \$9 profit.
   - A batch of loaves of bread requires 1 hour of work, 3kg flour, and yields \$10 profit.

   The bakery has 40 hours of labor and 100kg of flour available each day.
   Use this data to answer (a) - (e) below.

   (a) Using x_1, x_2, x_3 (respectively) to be the number of batches of baguettes, crackers, and loaves of bread made daily and adding slack variables as needed, write a linear program in *equational form* to find the maximum profit of the bakery per day.

$$\max 7x_1 + 9x_2 + 10x_3 \quad \text{s.t} \quad \mathbf{x} \geq 0, \quad \begin{aligned} x_1 + x_2 + x_3 + x_4 &= 40 \\ x_1 + 2x_2 + 3x_3 + x_5 &= 100 \end{aligned}$$

   (b) To maximize their profit every day, the bakery makes no baguettes, 20 batches of crackers, and 20 batches of loaves of bread. Give the basis and simplex tableau corresponding to this solution.

   Basis $B = \{2, 3\}$,
   $\mathcal{T}(B) =$

$$\begin{aligned} x_2 &= 20 - 2x_1 - 3x_4 + x_5 \\ x_3 &= 20 + x_1 + 2x_4 - x_5 \\ z &= 380 - x_1 - 7x_4 - x_5 \end{aligned}$$

(c) The bakery is considering increasing the number of hours worked each day. What is the maximum number of hours they can add per day so that they still maximize profits by making no baguettes and using up all of their available resources?

Answer: 10 hours

Justification: Basis $B = \{2, 3\}$, $A_B = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \to A_B^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$

$$A_B^{-1} \begin{pmatrix} 40+t \\ 100 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 40+t \\ 100 \end{pmatrix} = \begin{pmatrix} 20+3t \\ 20-2t \end{pmatrix} \geq 0 \quad \Leftrightarrow \quad -20/3 \leq t \leq 10$$

(d) If they add fewer hours than your answer from (c), how much additional profit can they make for every extra hour added?
Answer: \$7 for every hour added

Justification 1: Objective function is given by

$$\mathbf{c}_B^T A_B^{-1} \mathbf{b} = \begin{pmatrix} 9 & 10 \end{pmatrix} \begin{pmatrix} 20+3t \\ 20-2t \end{pmatrix} = 380 + 7t$$

Justification 2: The Dual LP $\min \mathbf{b}^T \mathbf{y}$ s.t. $A^T \mathbf{y} - \mathbf{s} = \mathbf{c}$, $\mathbf{s} \geq 0$ has optimal solution $(\mathbf{y}^*, \mathbf{s}^*)$, where $s_2^* = s_3^* = 0$, by complimentary slackness. This gives the system of linear equations

$$y_1 + y_2 - s_1 = 7, \; y_1 + 2y_2 = 9, \; y_1 + 3y_2 = 10, \; y_1 - s_4 = 0, \; y_2 - s_5 = 0$$

The second and third equations give $(y_1^*, y_2^*) = (7, 1)$. Since y_1 corresponds to the constraints on the number of hours, y_1^* gives the additional profit per hour.

(e) Starting from the original data, what is the minimum *profit per batch of baguettes* for which there is a solution maximizing profits that involves making (a positive number of) baguettes?

Answer: \$8 profit per batch (or an additional \$1 profit).

Justification: From the last line $z = 380 - x_1 - 7x_4 - x_5$ in the simplex tableau for $B = \{2, 3\}$, we see that

$$\begin{pmatrix} -1 & -7 & -1 \end{pmatrix} = \mathbf{c}_N^T - \mathbf{c}_B^T A_B^{-1} A_N = \begin{pmatrix} 7 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 9 & 10 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Increasing c_1 by t increased the coefficient of x_1 by t. If $t < 1$, the basis $B = \{2, 3\}$ is still (uniquely) optimal.

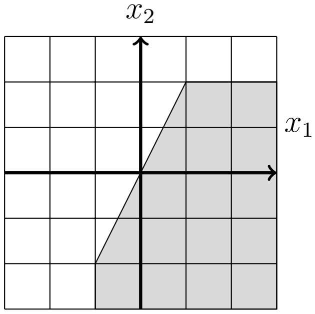
For $t > 1$, this basis is no longer optimal. We would pivot to find new optimal solution:

$$\begin{aligned} x_1 &= 1/2(20 - x_2 - 3x_4 + x_5) \\ x_3 &= 1/2(60 - x_2 + x_4 - x_5) \\ z &= 1/2(740 + 60t + (1 - t)x_2 + (-11 + t)x_4 + (-3 - t)x_5) \end{aligned}$$

4. (12 points) Consider the linear program

$$\max\ \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad x_2 \le 2x_1,\ \ x_1 \ge -1,\ \text{and}\ x_2 \le 2,$$

some of whose feasible region is shown below.



In each part below, **circle all** choices of $\mathbf{c}^T$ that make the linear program or its dual have the requested property.

Do not justify your answers.

(a) The linear program attains its optimal value at $(x_1, x_2) = (1, 2)$.

$\mathbf{c}^T = (-2, 1)$ $\mathbf{c}^T = (-1, 1)$ $\mathbf{c}^T = (0, 1)$ $\mathbf{c}^T = (1, 1)$ $\mathbf{c}^T = (1, 0)$

(b) The linear program has more than one optimal solution.

$\mathbf{c}^T = (-2, 1)$ $\mathbf{c}^T = (-1, 1)$ $\mathbf{c}^T = (0, 1)$ $\mathbf{c}^T = (1, 1)$ $\mathbf{c}^T = (1, 0)$

(c) The dual linear program is infeasible.

$\mathbf{c}^T = (-2, 1)$ $\mathbf{c}^T = (-1, 1)$ $\mathbf{c}^T = (0, 1)$ $\mathbf{c}^T = (1, 1)$ $\mathbf{c}^T = (1, 0)$

(d) The dual linear program is unbounded (from below).

$\mathbf{c}^T = (-2, 1)$ $\mathbf{c}^T = (-1, 1)$ $\mathbf{c}^T = (0, 1)$ $\mathbf{c}^T = (1, 1)$ $\mathbf{c}^T = (1, 0)$

5. (18 points) In each part below, circle True or False. Do not justify your answers.

(a) A linear program is feasible if and only if its auxiliary linear program is bounded.

True False

(b) Every polyhedron has a vertex.

True False

(c) If $\mathbf{v}$ is a point in a polyhedron P and $\mathbf{v}$ is not a vertex of P, then there exists a nonzero vector $\mathbf{w} \in \mathbb{R}^n$ and $\varepsilon > 0$ so that both $\mathbf{v} + \varepsilon\mathbf{w}$ and $\mathbf{v} - \varepsilon\mathbf{w}$ belong to P.

True False

(d) If $\mathbf{c}^T\mathbf{x}$ and $\tilde{\mathbf{c}}^T\mathbf{x}$ both achieve their maximum over a polyhedron P at $\mathbf{x}^*$, then for any $\lambda \geq 0$, $\mu \geq 0$, the maximum of $(\lambda\mathbf{c} + \mu\tilde{\mathbf{c}})^T\mathbf{x}$ over P is also achieved at $\mathbf{x}^*$.

True False

(e) A subset $B \subseteq \{1, \ldots, n\}$ is a feasible basis of $\max \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$ with $A \in \mathbb{R}^{m \times n}$ if and only if the matrix A_B is invertible.

True False

(f) Using Bland's pivot rule, the simplex method can return to previously visited basic feasible solutions.

True False

(g) If $\mathbf{c}^T$ can be written as a nonnegative linear combination of rows of A, then $\mathbf{c}^T\mathbf{x}$ bounded from above on $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$.

True False

(h) A linear program attains an optimal value if and only if its dual linear program attains the same optimal value.

True False

(i) If a linear program is infeasible, then its dual linear program is unbounded.

True False