Math 407 – Midterm
Autumn 2022
Solutions

Name: ______________________________

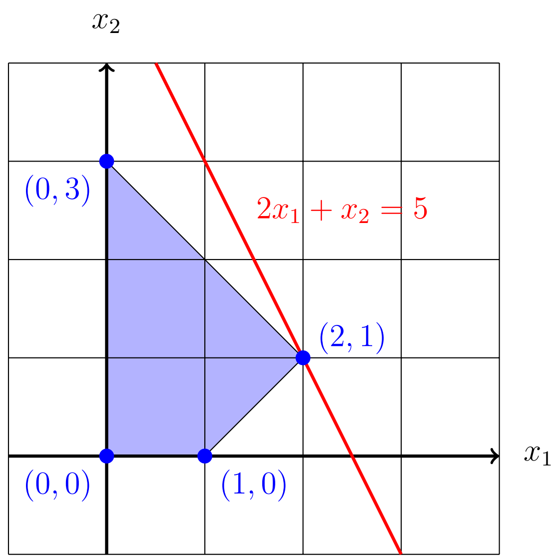
Student ID #: ______________________________

Question	Points	Score
1	18	
2	10	
3	10	
4	12	
5	10	
Total:	60	

1. (18 points) Parts (a) - (c) refer to the following linear program:

$$\max \quad 2x_1 + x_2 \quad \text{such that} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_1 + x_2 \leq 3, \quad x_1 - x_2 \leq 1, \quad x_1 \leq 2$$

(a) Plot the feasible region of this linear program and solve it graphically.
(An appropriately drawn and labeled line in your picture can serve as justification.)



optimal value $= 5$

optimal solution $= (2,1)$

(b) Give a matrix A and vectors $\mathbf{b}$, $\mathbf{c}$ so that $\max \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$ is a reformulation of this linear program.

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(c) List the vertices of the feasible region of the linear program in equational form and, for each vertex, list *every* feasible basis that corresponds to it.
(Use your picture in (a) to determine the vertices of the feasible region.)

vertex: $(0, 0, 3, 1, 2)$, basis: $\{3, 4, 5\}$

vertex: $(0, 3, 0, 4, 2)$, basis: $\{2, 4, 5\}$

vertex: $(2, 1, 0, 0, 0)$, bases: $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$

vertex: $(1, 0, 2, 0, 1)$, basis: $\{1, 3, 5\}$

2. (10 points) Here is a simplex tableau of an auxiliary linear program:

$$\begin{aligned} x_1 &= 1 + x_2 + x_3 + x_5 - x_7 \\ x_6 &= 2 - x_2 + x_3 + x_4 \\ z &= -2 + x_2 - x_3 - x_4 - x_7 \end{aligned}$$

(a) What is the number of variables and number of equations in the original linear program? (Do not justify your answer.)

\# variables = 5 \# equations = 2

(b) Perform a pivot step on the tableau above and write down the resulting feasible basis and new simplex tableau.

Entering variable: x_2

Tightest bound coming from $x_6 = 2 - x_2 \geq 0 \Rightarrow x_6$ leaves.

New basis: $\{1, 2\}$.

Simplex tableau:

$$\begin{aligned} x_1 &= 3 + 2x_3 + x_4 + x_5 - x_6 - x_7 \\ x_2 &= 2 + x_3 + x_4 - x_6 \\ z &= -x_6 - x_7 \end{aligned}$$

(c) Is the original linear program feasible? If no, explain why. If yes, provide a basic feasible solution of the original linear program.

Because the auxiliary linear program has optimal value equal to zero, the original linear program is equal to zero.

Basic feasible solution: $(3, 2, 0, 0, 0)$

(attained from dropping x_6, x_7 from optimal basic feasible solution of Aux LP)

3. (10 points) Here is a simplex tableau of a linear program with a mystery parameter λ in the last row.

$$\begin{aligned} x_1 &= 9 + x_4 + x_5 - 2x_6 \\ x_2 &= 5 + x_5 - x_6 \\ x_3 &= 3 - x_6 \\ z &= 8 + \lambda x_5 - 2x_6 \end{aligned}$$

(a) For what values of $\lambda \in \mathbb{R}$ is this linear program bounded? (Justify your answer.)

For $\lambda \leq 0$, this linear program is bounded. Then all coefficients of variables in z are non-positive, which means that the optimal value is $z_0 = 8$.

If $\lambda > 0$, then we can pivot so that x_5 enters. We see that there is no upper bound on x_5 and that the point $\mathbf{x} = (9+x_5, 5+x_5, 3, 0, x_5, 0)$ is feasible for all $x_5 \geq 0$ with objective function value $8 + \lambda x_5 \to \infty$ as $x_5 \to \infty$.

(b) If λ satisfies your answer from (a), what is the optimal value and optimal solution in $\mathbb{R}^6$ of this linear program?

For $\lambda \leq 0$, all coefficients of variables in z are non-positive, meaning that the basis $\{1, 2, 3\}$ is optimal.

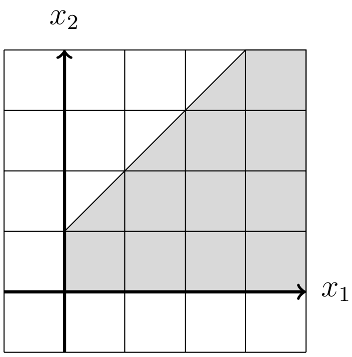
Optimal value $= 8$

Optimal solution $= \mathbf{x} = (9, 5, 3, 0, 0, 0)$

4. (12 points) Consider the linear program

$$\max\ \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_2 - x_1 \leq 1$$

whose feasible region is shown below.



In each part below, **circle all** choices of $\mathbf{c}^T$ that make the linear program have the requested property.

Do not justify your answers.

(a) The linear program is unbounded.

$\mathbf{c}^T = (-2, 1)$ $\mathbf{c}^T = (-1, 0)$ $\mathbf{c}^T = (-1, 1)$ $\mathbf{c}^T = (1, -1)$ $\mathbf{c}^T = (1, 3)$

(b) The linear program is feasible.

$\mathbf{c}^T = (-2, 1)$ $\mathbf{c}^T = (-1, 0)$ $\mathbf{c}^T = (-1, 1)$ $\mathbf{c}^T = (1, -1)$ $\mathbf{c}^T = (1, 3)$

(c) The linear program attains its optimal value at $\mathbf{x} = (0, 1)$.

$\mathbf{c}^T = (-2, 1)$ $\mathbf{c}^T = (-1, 0)$ $\mathbf{c}^T = (-1, 1)$ $\mathbf{c}^T = (1, -1)$ $\mathbf{c}^T = (1, 3)$

(d) The linear program has more than one optimal solution.

$\mathbf{c}^T = (-2, 1)$ $\mathbf{c}^T = (-1, 0)$ $\mathbf{c}^T = (-1, 1)$ $\mathbf{c}^T = (1, -1)$ $\mathbf{c}^T = (1, 3)$

5. (10 points) In each part below, circle True or False. Do not justify your answers.

   (a) Every nonempty polyhedron of the form $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ has a vertex.

   True False

   (b) A linear program is bounded if and only if its feasible region is bounded.

   True False

   (c) The set of points achieving the optimal value of a bounded linear program is convex.

   True False

   (d) If $\mathbf{v}$ and $\mathbf{w}$ are basic feasible solutions of a linear program, then so is $\lambda\mathbf{v} + (1-\lambda)\mathbf{w}$ for all $0 \leq \lambda \leq 1$.

   True False

   (e) Every basic feasible solution of a linear program $\max \mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$ with $A \in \mathbb{R}^{m \times n}$ has exactly $n - m$ coordinates equal to zero.

   True False