Math 407

Review

LP Duality: search for best upper bound on an linear program explicitly implied by constraints

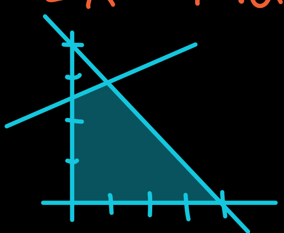
(P) $\max c^T x$ s.t. $Ax \le b$ $\to$ (D) $\min b^T y$ s.t. $y \ge 0$, $A^T y = c$

(P) $\max c^T x$ s.t. $Ax = b$, $x \ge 0$ $\to$ (D) $\min b^T y$ s.t. $A^T y - s = c$, $s \ge 0$

Any feasible pt y of (D) gives an upper bound $b^T y$ on (P) and any feasible pt x of (P) gives a lower bound of $c^T x$ on (D).

Ex: $\max x_2$ s.t. $x_1 + x_2 + x_3 = 4$

$-x_1 + 2x_2 + x_4 = 5$ $\quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \ge 0$



$\Rightarrow y_1(x_1 + x_2 + x_3) + y_2(-x_1 + 2x_2 + x_4)$

$- s_1 x_1 - s_2 x_2 - s_3 x_3 - s_4 x_4 \le 4y_1 + 5y_2$

holds on any feas. pt region of (P) for any $y \in \mathbb{R}^2$, $s \ge 0$

(D) $\min 4y_1 + 5y_2$ s.t.

$s_1 \ge 0$, $s_2 \ge 0$, $s_3 \ge 0$, $s_4 \ge 0$

$y_1 - y_2 - s_1 = 0 \iff y_1 - y_2 = s_1 \ge 0$

$y_1 + 2y_2 - s_2 = 1 \iff y_1 + 2y_2 = 1 + s_2 \ge 1$

$y_1 - s_3 = 0 \iff y_1 = s_3 \ge 0$

$y_2 - s_4 = 0 \iff y_2 = s_4 \ge 0$

Equivalent formulation:

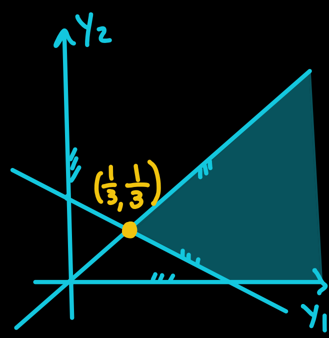
$\min 4y_1 + 5y_2$ s.t.

$y_1 - y_2 \ge 0$,

$y_1 + 2y_2 \ge 1$

$y_1 \ge 0$

$y_2 \ge 0$



Strong Duality Thm: The only possibilities are

(1) both (P), (D) infeasible

(2) (P) unbounded, (D) infeasible

(3) (P) infeasible, (D) unbounded

(4) (P), (D) both feasible and bounded and have equal optimal values

Optimality: If x^* is feasible for (P) and y^* is feasible (D) then the following are equivalent

1) x^* is optimal for (P) and y^* is optimal for (D)

2) $c^T x^* = b^T y^*$

3) x^*, y^* satisfy complimentary slackness

e.x. (P) max $c^T x$ s.t. $Ax \le b$ (D) min $b^T y$ s.t. $A^T y = c,\ y \ge 0$

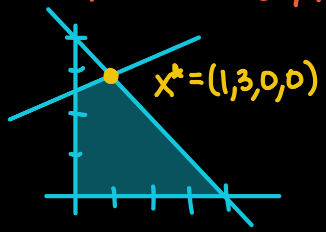
Comp Slack: $(b_i - a_i^T x_i^*) y_i^* = 0$ for all $i = 1, \dots, m$

e.x. (P) max $c^T x$ s.t. $Ax = b,\ x \ge 0$ (D) min $b^T y$ s.t. $A^T y - s = c,\ s \ge 0$

Comp Slack: $s_i^* x_i^* = 0$ for all $i = 1, \dots, n$

equiv $(A_i^T y^* - c_i) x_i^* = 0$

Ex: max x_2 s.t. $x_1 + x_2 + x_3 = 4$, $-x_1 + 2x_2 + x_4 = 5$, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geq 0$



$(y_1, y_2) = (\frac{1}{3}, \frac{1}{3})$ $\quad s^T = (0, 0, 1/3, 1/3)$

$$\frac{1}{3}(x_1 + x_2 + x_3 = 4) + \frac{1}{3}(-x_1 + 2x_2 + x_4)$$
$$+ 0(-x_1 \leq 0) + 0(-x_2 \leq 0) + \frac{1}{3}(-x_3 \leq 0) + \frac{1}{3}(-x_4 \leq 0)$$

$\Rightarrow x_2 \leq \frac{4}{3} + \frac{5}{3} = 3$ best possible upper bound

(2) $c^T x^* = 3 = b^T y^* \Rightarrow x^*$ optimal for (P), y^* optimal for (D)

(3) Comp. Slackness $\quad x^* = (1, 3, 0, 0)$ in each coordinate
$\quad s^* = (0, 0, 1/3, 1/3)$ $\quad x_i^* = 0$ or $s_i^* = 0$
$\quad \uparrow \uparrow \uparrow \uparrow$

$\Rightarrow x^*$ optimal for (P), y^* optimal for (D)

How would you find (y^*, s^*) from x^*?

By complimentary slackness, any optimal dual solution have $s_1^* = s_2^* = 0$ (and be feasible for (D)).

Feasibility for (D) $\Rightarrow$
$$\left.\begin{aligned} y_1 - y_2 - s_1 &= 0 \\ y_1 + 2y_2 - s_2 &= 1 \\ y_1 - s_3 &= 0 \\ y_2 - s_4 &= 0 \end{aligned}\right\}$$
Together with $s_1 = s_2 = 0$ this has a unique sol: $y_1 = y_2 = s_3 = s_4 = \frac{1}{3}$ $\quad s_1 = s_2 = 0$