Math 407

Sensitivity Analysis
(Recap & new variables or constraints)

(P) $\max c^T x$ s.t. $Ax=b$, $x \geq 0$

Suppose basis $B \subseteq \{1,\dots,n\}$ is feasible $(p \geq 0)$ and dual feasible $(r \leq 0)$ $\Rightarrow$ optimal

Changing $c \to \tilde{c}$: B always still feasible

remains optimal $\iff$ $\tilde{c}_N^T - \tilde{c}_B^T A_B^{-1} A_N \leq 0$.

no longer optimal $\to$ pivot from B in simplex method

Changing $b \to \tilde{b}$: B remains dual feasible

remains feasible $\iff$ $A_B^{-1} b \geq 0$

no longer feasible $\to$ pivot from B in dual simplex method

Adding a new variable

(NP) $\max c^T x + c_{new} x_{new}$ s.t. $Ax + a_{new} x_{new} = b$, $\begin{pmatrix} x \\ x_{new} \end{pmatrix} \geq 0$

Note: B still feasible and $x_{new} \to 0$ recovers original problem

$T(B)$ for (NP):

$$x_B = p + Q x_N - A_B^{-1} a_{new} x_{new}$$
$$z = z_0 + r^T x_N + \underline{(c_{new} - c_B^T A_B^{-1} a_{new})}\, x_{new}$$

feasibility ($p \geq 0$) unchanged

B still optimal $\iff$ $c_{new} - c_B^T A_B^{-1} a_{new} \leq 0$

no longer optimal $\rightarrow$ pivot from B in simplex method

Ex: Our knitting store is considering also making mittens. One pair requires 5 hours of work and 100g of yarn. How much profit should be made per pair for it to be worth making them?

c_{new} = profit/pair of mittens

x_{new} = # pairs produced

(NP) max $10x_1 + 6x_2 + c_{new} x_{new}$ s.t.

$$6x_1 + 4x_2 + x_3 + \boxed{5}\, x_{new} = 80$$
$$200x_1 + 100x_2 + x_4 + \boxed{100}\, x_{new} = 2500$$

a_{new}

$\begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix} \geq 0 \quad x_{new} \geq 0$

$$c_B^T A_B^{-1} a_{new} = (10\ \ 6)\begin{pmatrix} 6 & 4 \\ 200 & 100 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 100 \end{pmatrix} = 7$$

T(B):
$$x_1 = 10 + \frac{x_3}{2} - \frac{x_4}{50} + \frac{x_{new}}{2}$$
$$x_2 = 5 - x_3 + \frac{3x_4}{100} - 2x_{new}$$
$$z = 130 - x_3 - \frac{x_4}{50} + \boxed{(c_{new} - 7)}\, x_{new}$$

If profit/pair of mittens is < \$7, $c_{new} - 7 < 0$

$\Rightarrow$ optimal solution is 10 scarves, 5 hats, 0 mittens

If profit/pair of mittens is > \$7, $c_{new} - 7 > 0$,

$\Rightarrow$ no longer optimal! Pivot to find new solution e.g. $c_{new} = 8 \rightarrow$ x_{new} enters, x_2 leaves

$$x_1 = 45/4 - x_2/4 + x_3/4 - x_4/80$$
$$x_{new} = 5/2 - x_2/2 - x_3/2 + 3x_4/200$$
$$z = 265/2 - x_2/2 - 3x_3/2 - x_4/200$$

Profit of \$132.5 from 11.25 scarves and 2.5 mittens / week.

Adding a new constraint

(NP) $\max\ c^T x$ s.t. $Ax = b$, $x \geq 0$, $\underbrace{a_{new}^T x \leq b_{new}}$

$\hookrightarrow a_{new}^T x + x_{new} = b$

$\hookrightarrow \tilde{A} = \begin{pmatrix} A \\ a_{new}^T \end{pmatrix}$ $\tilde{b} = \begin{pmatrix} b \\ b_{new} \end{pmatrix}$ $\tilde{c}^T = (c^T, 0)$

Given basis $B \subseteq \{1, \dots, n\}$, consider $\tilde{B} = B \cup \{n+1\}$.

$$\tilde{A}_{\tilde{B}} = \begin{pmatrix} A_B & 0 \\ (a_{new})_B^T & 1 \end{pmatrix} \Rightarrow \tilde{A}_{\tilde{B}}^{-1} = \begin{pmatrix} A_B^{-1} & 0 \\ -(a_{new})_B^T & 1 \end{pmatrix}$$

$T(\tilde{B})$:
$$x_B = p + Qx_N$$
$$x_{n+1} = \boxed{} + \boxed{}\, x_N$$
$$z = z_0 + r^T x_N$$

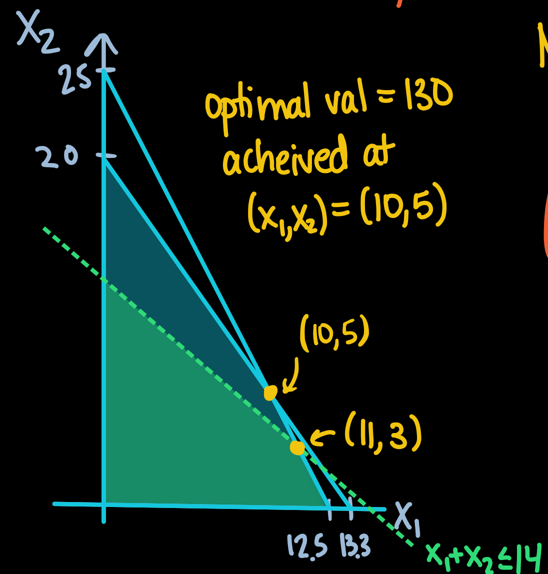
$b_{new} - (a_{new})_B^T b$

B still dual feasible

B still feasible $\iff b_{new} - (a_{new})_B^T b \geq 0$

Otherwise pivot in dual simplex method!

Ex: Our knitting store adds tags to their items and only has 14 tags available!



New constraint: $x_1+x_2\le 14 \rightarrow x_1+x_2+x_5=14$
$x_5\ge 0$

(NP) $\max\ 10x_1+6x_2$ s.t.

$$\begin{aligned} 6x_1+4x_2+x_3&=80\\ 200x_1+100x_2+x_4&=2500\\ x_1+x_2+x_5&=14 \end{aligned}, \quad \begin{pmatrix} x_1\\ \vdots \\ x_5\end{pmatrix}\ge 0$$

For (OP), basis $\{1,2\}$ was feasible and dual feasible

$\Rightarrow$ For (NP), basis $\{1,2,5\}$ is dual feasible

$T(\{1,2,5\})$

$$\begin{aligned} x_1&=10+\tfrac{1}{2}x_3-\tfrac{1}{50}x_4\\ x_2&=5-x_3+\tfrac{3}{100}x_4\\ x_5&=\boxed{-1}+\tfrac{1}{2}x_3-\tfrac{1}{100}x_4\\ z&=130-x_3-\tfrac{1}{50}x_4 \end{aligned}$$

dual feasible but not feasible
$\Rightarrow$ dual pivot step!
x_5 leaves, x_3 enters

$T(\{1,2,3\})$

$$\begin{aligned} x_1&=11-\tfrac{1}{100}x_4+x_5\\ x_2&=3+\tfrac{1}{100}x_4-2x_5\\ x_3&=2+\tfrac{1}{50}x_4+2x_5\\ z&=128-\tfrac{1}{25}x_4-2x_5 \end{aligned}$$

only one dual pivot to recover feasibility!
new opt sol:
11 scarves, 3 hats
(and 2 unused worker hours)

profit: \$128