Math 407

Shadow prices and economic interpretations

A company makes n products from m resources.

c_j = profit/unit of j^{th} product

b_i = amount of resource i available

a_{ij} = amount of resource i used in production of one unit of product j.

Maximizing profits as an LP:

$$(P) \quad \max \sum_{j=1}^{n} c_j x_j \quad \text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i=1,\dots,m$$
$$x_j \geq 0 \quad j=1,\dots,n$$

where x_j = amount of resource j produced

With more of a given resource, the company could produce more and profit more.

Q: How much more should they be willing to pay (above trading price) for more of resource i?

Note: If x^* is an optimal solution of (P) and $a_i^T x^* < b_i$, there is already extra resource i available $\Rightarrow$ having a small amt more won't help.

If $a_i^T x^* = b_i$, it might. The dual LP is

$$\min b^T y \text{ s.t. } c_j \leq \sum_{i=1}^m a_{ij} y_i,\ j=1,\dots,n,\quad y_i \geq 0,\ i=1,\dots,m$$

Suppose $y^* \in \mathbb{R}^m$ is the unique solution of (D). Then for any $i=1,\dots,m$, there is some $\varepsilon > 0$, so that y^* also minimizes $(b+te_i)^T y$ for all $0 \leq t \leq \varepsilon$.

Since primal and dual solutions are equal, the max profit with an additional t units of resource i is

$$(b+te_i)^T y^* = b^T y^* + t e_i^T y^* = \underbrace{b^T y^*}_{\text{previous profit}} + \underbrace{t y_i^*}_{\text{additional profit}}$$

Paying an additional d \$/unit of resource i costs d \$/unit but yields y_i^* \$/unit in profit. If $d < y_i^*$, this will increase net profits.

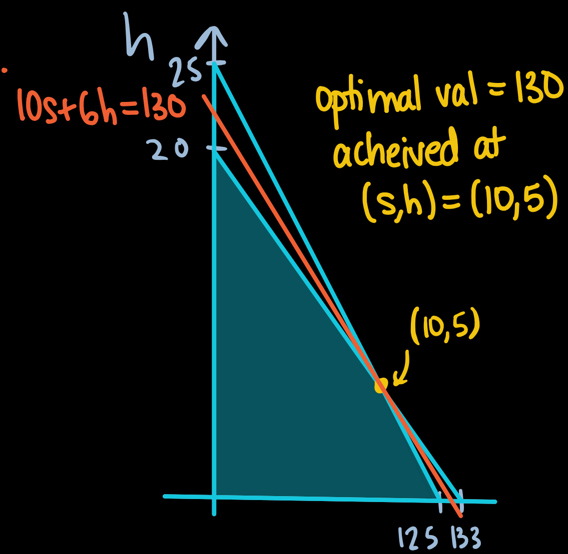
additional profit (without rate hike) : \$ $t y_i^*$

extra cost : \$ $t d$

net gain : \$ $t(y_i^* - d)$

The value y_i^* is called the <u>shadow price</u> or <u>marginal price</u> for resource i.

Ex: A small business makes hats and scarves from imported wool. Scarves require 6 hours of work, 200g of wool, and yield \$10 profit. Hats require 4 hours of work, 100g of wool and yield a profit of \$6.



max $10s+6h$ s.t.

$s \geq 0,\ h \geq 0,$

$6s+4h \leq 80,\ 200s+100h \leq 2500$

If the company paid overtime (higher wages for extra hours), they would have more hours available.

The optimal dual solution is $y_1^* = 1,\ y_2^* = 1/50$.

$\Rightarrow$ profit $= b^T y^* = 80 \cdot y_1^* + 2500 y_2^* = 130$

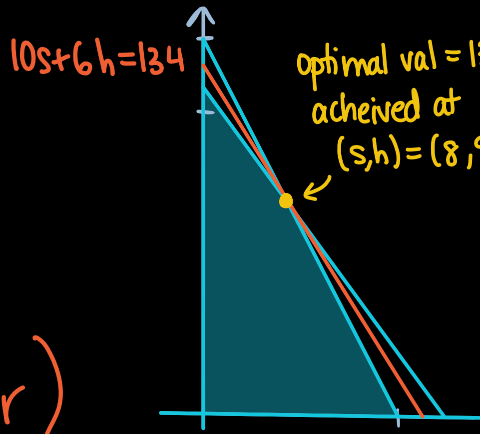
For small enough changes in b,

$$\text{profit} = (b+\Delta b)^T y^* = (80+t_1)y_1^* + (2500+t_2)y_2^*$$
$$= 130 + t_1 + \tfrac{1}{50}t_2$$

The increase in profit is $y_1^* t_1 + y_2^* t_2 = t_1 + \frac{1}{50}t_2$.

If they pay less than \$1/hr to get an additional work hour, their profit will increase.

Ex: Pay extra 50 cents/hr to get an addition 4 hours / week.

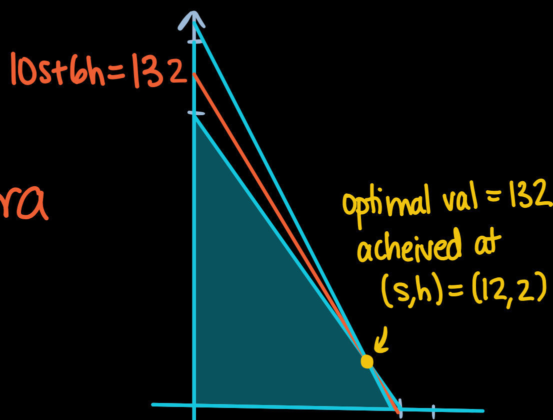


(e.g. Base salary \$15/hr,
OT salary (only for extra hrs) : \$15.50/hr)

Costs an extra \$2 but will bring in an extra \$4

Net gain: $\$4-\$2=\$2$

Similarly they should pay less than $1/50$ \$/gram extra to gain additional wool.



Ex: Extra 100g wool for $\$\frac{1}{100}/g$ extra

Costs \$1, brings in extra \$2

Net gain: $\$2-\$1=\$1$

Note: If $\Delta b_2 > 500/3 \approx 166.67$ the solution (s,h) to $6s+4h=80$, $200s+100h=2500+\Delta b_2$ is no longer feasible ($\Rightarrow$ can't be optimal)

By complimentary slackness the optimal dual solution also changes (no longer $(y_1^*, y_2^*) = (1, 1/50)$)

$\Rightarrow$ not guaranteed an additional $\Delta b_2 y_2^*$ in profit.