Math 407

Today: Sensitivity Analysis for Tableaux

In equational form: $\max c^T x$ s.t. $Ax=b$, $x \geq 0$

Recall, the simplex tableau of a feasible basis B is

$$x_B = p + Q x_N \qquad p = A_B^{-1} b \qquad Q = -A_B^{-1} A_N$$

$$z = z_0 + r^T x_N \qquad z_0 = c_B^T A_B^{-1} b \qquad r^T = c_N^T - c_B^T A_B^{-1} A_N$$

Same sol as $A_B x_B + A_N x_N = b$, $z = c_B^T x_B + c_N^T x_N$

$\Rightarrow$ $x_B = A_B^{-1} b - A_B^{-1} A_N x_N$,

$z = c_B^T (A_B^{-1} b - A_B^{-1} A_N x_N) + c_N^T x_N$

Recall: B corresponds to the point given by $Ax=b$, $x_N = 0$

namely $x_B = A_B^{-1} b = p$, $x_N = 0$

z_0 = value of $c^T x$ at this point.

Similary in the dual space B corresponds to the solution of $A^T y - s = c$ with $s_B = 0$,

namely $s_B = 0$ $s_N = -r$, y = solution to $A^T y = s + c$

z_0 = value of $b^T y$ at this point

Suppose a basis B is feasible $(p \geq 0)$ and dual feasible $(r \leq 0)$

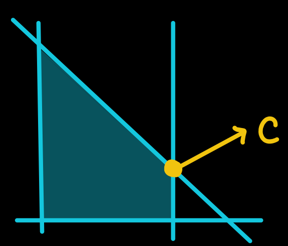
<u>Changing $c \to \tilde{c}$</u> : only r changes

$\Rightarrow$ B always remains feasible

B remains dual feasible $\Leftrightarrow$ $\tilde{c}_N^T - \tilde{c}_B^T A_B^{-1} A_N \leq 0$.

<u>Reasonable hope</u>: If B is no longer dual feasible after changing $c \to \tilde{c}$, then it should only be a few pivot steps away from the (new) optimal basis (if the problem remains bounded)

Ex: $\max\ 5x_1 + 4x_2$ s.t. $x_1 + x_2 + x_3 = 3$, $x_1 + x_4 = 2$, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geq 0$



$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ \underline{1} & \underline{0} & 0 & 1 \end{pmatrix}$ $\quad b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\quad c = \begin{pmatrix} 5 \\ 4 \\ 0 \\ 0 \end{pmatrix}$

Opt sol: $x^* = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad A_B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad A_N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Opt. basis: $\{1, 2\}$ $\quad A_B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad c_B = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ $\quad c_N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

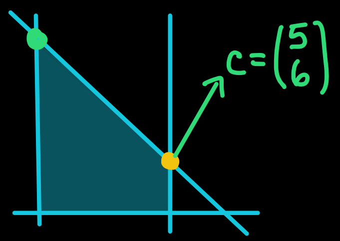
$\tilde{c} = c + te_2$ $\quad \tilde{c}_B = \begin{pmatrix} 5 \\ 4+t \end{pmatrix}$ $\quad \tilde{c}_N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\tilde{c}_N^T - \tilde{c}_B^T A_B^{-1} A_N = -(5 \quad 4+t) \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = (-4-t, -1+t)$

$B=\{1,2\}$ remains dual feasible $\Leftrightarrow -4 \leq t \leq 1$

$$T(\{1,2\}) \quad \begin{aligned} x_1 &= 2 && && - x_4 \\ x_2 &= 1 && - x_3 && + x_4 \\ z &= 14+t && -(4+t)x_3 && +(-1+t)x_4 \end{aligned}$$

$t=2$



$$T(\{1,2\}) \quad \begin{aligned} x_1 &= 2 && && - x_4 \\ x_2 &= 1 && - x_3 && + x_4 \\ z &= 16 && - 6x_3 && \underline{\underline{+}}\, x_4 \end{aligned}$$

no longer dual feasible
$\to$ pivot!
x_4 enters, x_1 leaves

$$T(\{2,4\}): \quad \begin{aligned} x_2 &= 3 - x_1 - x_3 \\ x_4 &= 2 - x_1 \\ z &= 18 - x_1 - 6x_3 \end{aligned}$$

new opt sol: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 2 \end{pmatrix}$

<u>Changing $b \to \tilde{b}$</u>: only p changes

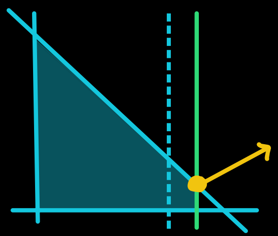
$\Rightarrow$ B always remains dual feasible

B remains feasible $\Leftrightarrow A_B^{-1} b \geq 0$

<u>Reasonable hope</u>: If B is no longer feasible after changing $b \to \tilde{b}$, then it should only be a few dual pivot steps away from the (new) optimal basis (if the problem remains feasible)

Ex: max $5x_1+4x_2$ s.t. $x_1+x_2+x_3=3$ $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\geq 0$

$x_1+x_4=2+t$



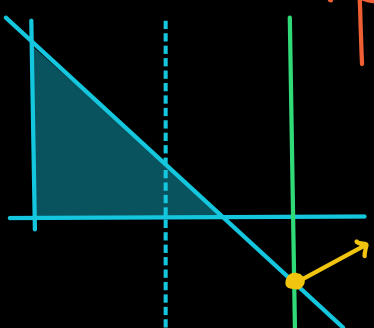
$B=\{1,2\}$ remains dual feasible

$$\Leftrightarrow \quad 0\leq A_B^{-1}\tilde{b}=\begin{pmatrix}0&1\\1&-1\end{pmatrix}\begin{pmatrix}3\\2+t\end{pmatrix}=\begin{pmatrix}2+t\\1-t\end{pmatrix}$$

$$\Leftrightarrow \quad -2\leq t\leq 1$$

$T(\{1,2\})$:
$$x_1 = 2+t \qquad\qquad - x_4$$
$$x_2 = 1-t - x_3 + x_4$$
$$z = \underbrace{14+t}_{\text{= val of } c^Tx \text{ at } x=(2+t,1-t,0,0)} - 4x_3 - x_4$$

$t=2$



$T(\{1,2\})$:
$$x_1=4 \qquad -x_4$$
$$x_2=\underline{\underline{-1}}-x_3+x_4$$
$$z=16-4x_3-x_4$$

no longer feasible
→ dual pivot
x_2 leaves
x_4 enters

$T(\{1,4\})$:
$$x_1=3-x_2-x_3$$
$$x_4=1+x_2+x_3$$
$$z=15-x_2-5x_3$$

new opt solution: $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}3\\0\\0\\1\end{pmatrix}$

What happens for $t=-3$?