Math 407

Today: Intro to Sensitivity Analysis

Sensitivity Analysis

Often the values in constraints or objective function are estimated or change over time. Sensitivity analysis aims to understand how the solutions of an LP change with small changes in the problem data.

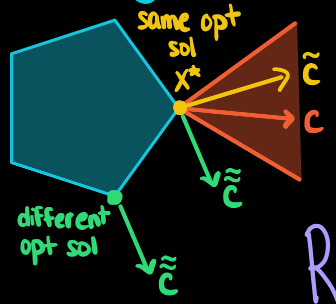
Original Primal (OP) max $c^T x$ s.t. $Ax \leq b$

New Primal (NP) max $\tilde{c}^T x$ s.t. $\tilde{A} x \leq \tilde{b}$

Questions of interest:

- How much can we change entries of A, b, c without changing the optimal solution / feasible basis?
- How do these changes affect the opt. value?
- How can we use the solution of (OP) to help solve (NP) (rather than having to start from scratch)?

Changing obj. function: $c \mapsto \tilde{c}$



Some changes in c do not change optimal solution. Some do!

Remark: If optimum solution x^* stays the same, optimal value changes linearly: $\underbrace{\tilde{c}^T x^*}$ linear in $\tilde{c}$ for fixed x^*

Prop: For a point x^* in $P = \{x \in \mathbb{R}^n : Ax \leq b\}$,

$\{c \in \mathbb{R}^n$ s.t. max $c^T x$ over P is attained at $x^*\}$

$= \{\sum_{i \in I} y_i a_i$ s.t. $y_i \geq 0 \ \forall i \in I\}$ where $I = \{i : a_i^T x^* = b_i\}$

(Proof by complimentary slackness!)

This is a convex cone (convex & closed under nonneg scaling)

Known as the normal cone of $\{x \in \mathbb{R}^n : Ax \leq b\}$ at x^*

Remark: If x^* is a vertex of $\{x \in \mathbb{R}^n : Ax \leq b\}$ then $\{a_i : i \in I\}$ spans $\mathbb{R}^n \Rightarrow$ normal cone has dim n.

If $\{a_i : i \in I\}$ form a basis for $\mathbb{R}^n$, then for any c, there is a unique solution $(y_i)_{i \in I}$ to $\underbrace{c = \sum_{i \in I} y_i a_i}$

square system
n eq in n variables $y_1, \dots, y_n$

The max$\{c^T x : x \in P\}$ is acheived at $x^* \iff$ unique sol has $y_i \geq 0 \ \forall i \in I$

Ex: Hats and scarves from week 1

\$10 profit per scarf and \$6 per hat

Variables: $s = \#$ scarves $\quad h = \#$ hats

Objective: maximize $10s + 6h$ (profit)

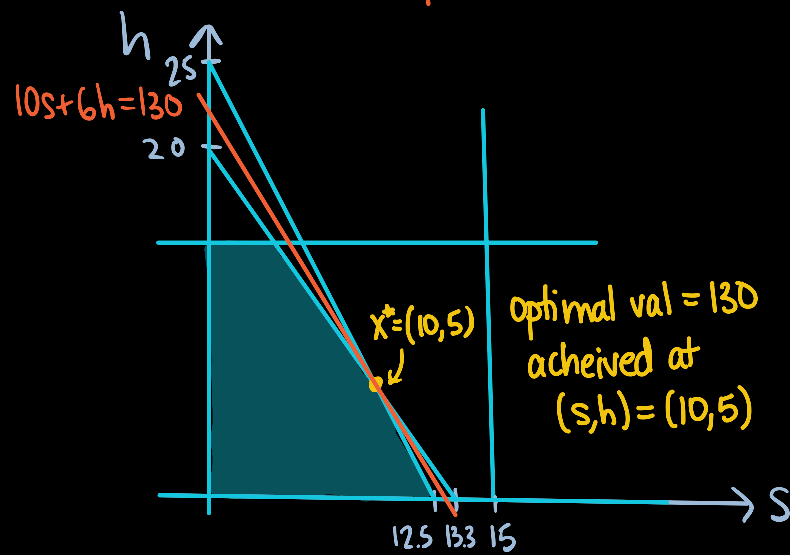
Constraints:

$$0 \le s \le 15$$

$$0 \le h \le 15$$

$$6s + 4h \le 80$$

$$200s + 100h \le 2500$$

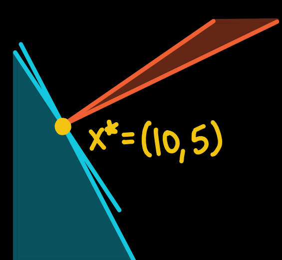


optimal val = 130 acheived at $(s,h) = (10,5)$

How sensitive to the price of scarves is this solution?

Active constraints at $x^* = (10,5)$:

$$6s + 4h \le 80 \qquad 200s + 100h \le 2500$$



Dual solution

$$y_5(6s+4h) + y_6(200s+100h) = 10s + 6h$$

$$\Longrightarrow \begin{pmatrix} y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2/100 \end{pmatrix} = \begin{pmatrix} 6 & 200 \\ 4 & 100 \end{pmatrix}^{-1} \begin{pmatrix} 10 \\ 6 \end{pmatrix}$$

Change price of scarves? Profit $10 \to 10+t$ \$/scarf

Same optimal solution $(s^*, h^*) = (10, 5)$

$$\iff \begin{pmatrix} 0 \\ 0 \end{pmatrix} \le \begin{pmatrix} y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} 6 & 200 \\ 4 & 100 \end{pmatrix}^{-1} \begin{pmatrix} 10+t \\ 6 \end{pmatrix} = \begin{pmatrix} 1 - t/2 \\ 1/50 + t/50 \end{pmatrix}$$

$$\iff -1 \le t \le 2$$

For $t \in [-1, 2]$, with \$$(10+t)$ per scarf, \$6 per hat, max profit is $(10+t)10 + 6 \cdot 5 = 130 + 10t$ dollars.

Certificate of optimality:

$$\begin{array}{rl} & (1 - t/2)(6s + 4h \le 80) \\ + & (1/50 + t/50)(200s + 100h \le 2500) \\ \hline & (10+t)s + 6h \le 130 + 10t \end{array}$$