Math 407

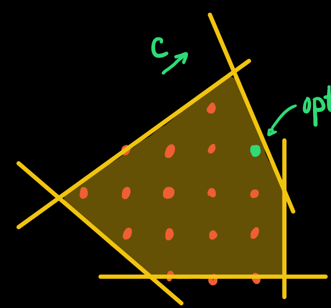
Today: Integer Programming

No class Friday: Happy Thanksgiving!

Integer programming

An integer program is a problem of the form

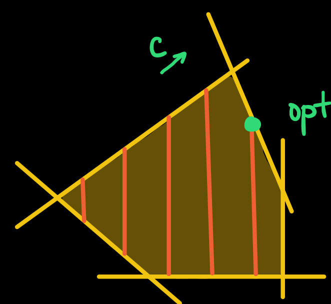
$$\max\ c^T x \quad \text{s.t.}\quad Ax \le b,\ x \in \mathbb{Z}^n$$



(maximize a linear function over the integer points $P \cap \mathbb{Z}^n$ of a polyhedron $P = \{x \in \mathbb{R}^n : Ax \le b\}$)

Also: mixed integer programming

$$\max\ c^T x \quad \text{s.t.}\quad Ax \le b,$$
$$x_1, \dots, x_k \in \mathbb{Z},\quad x_{k+1}, \dots, x_n \in \mathbb{R}$$



The constraint $x_i \in \mathbb{Z}$ occurs naturally in many applications (e.g. x_i = the number of...). So far we've ignored these constraints!

Unlike LP's, integer programs can be hard to solve both in $\underbrace{\text{theory}}_{\text{NP Hard}}$ and $\underbrace{\text{in pratice}}_{\text{10 ineq, 10 var out of reach computationally}}$.

Idea: If P is bounded, $P \cap \mathbb{Z}^n$ is finite.

List all pts $x \in P$ and compare $c^T x$

Trouble: this might be too many pts!

e.g. $P = \{x \in \mathbb{R}^n : 0 \le x_i \le 1, i = 1, \dots, n\}$ $|P \cap \mathbb{Z}^n| = 2^n$

Idea: Ignore constraints "$x \in \mathbb{Z}^n$"

LP Relaxation

ignoring "$x \in \mathbb{Z}^n$" constraint $\longrightarrow$ upper bound original problem

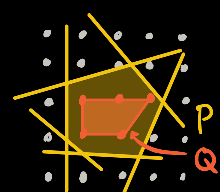
$$(\max c^T x \text{ s.t. } Ax \le b, x \in \mathbb{Z}^n) \le (\max c^T x \text{ s.t. } Ax \le b)$$

usually strict inequality!

Remark: If $P = \{x \in \mathbb{R}^n : Ax \le b\}$ is bounded then $Q = \text{conv}(P \cap \mathbb{Z}^n)$ is also a polyhedron and

$$(\max c^T x \text{ s.t. } x \in P \cap \mathbb{Z}^n) = (\max c^T x \text{ s.t. } x \in Q)$$

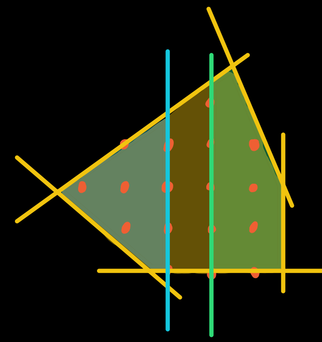
also an LP?



The problem is that we have to be able to compute an inequality description for Q efficiently

Idea: From some inequalities on P we can round to get better inequalities on Q.

Some general strategies:



- Branch and bound:

  Solve LP $\to$ opt solution x^*

  If $x_j^* \notin \mathbb{Z}$, solve new LP's with $x_j \le \lfloor x_j^* \rfloor$ vs. $x_j \ge \lceil x_j^* \rceil$

  one of the most common ways LPs are used in practice!

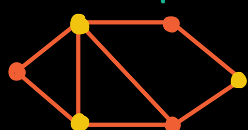
- Chvátal-Gomory cuts: for inequalities

  $a^T x \le b$ satisfied by all $x \in P$, $a \in \mathbb{Z}^n$, $b \in \mathbb{Q}$

  $\Rightarrow a^T x \le \lfloor b \rfloor$ satisfied by all $x \in P \cap \mathbb{Z}^n$

See also Gomory cuts (in glossary of book)

Example: Min Vertex Cover



vertex cover of size 3

no edges

$G = (V, E)$ a graph

$W \subseteq V$ is a vertex cover if every edge has at least one endpoint in W (i.e. $|e \cap W| \ge 1 \ \forall e \in E$)

Min Vertex Cover: $\min \{|W| : W \text{ is a vertex cover of } G\}$

This is NP-Hard to compute!

Or even to approximate within a factor of $4/3$!

As an integer program:

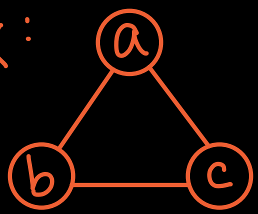
$$\min \sum_{v \in V} y_v \quad \text{s.t.} \quad 0 \le y_v \le 1 \ \forall v \in V, \quad y_v + y_w \ge 1 \ \forall \{v,w\} \in E$$
$$y_v \in \mathbb{Z} \ \forall v \in V$$

$y \in \mathbb{Z}^V$ feasible $\Longrightarrow$ $y_v \in \{0,1\}$ for all $v \in V$

Take $W = \{v \in V \text{ s.t. } y_v = 1\}$

$\Longrightarrow$ W is a vertex cover with $|W| = \sum_{v \in V} y_v$.

Ex:



IP: min $y_a + y_b + y_c$ s.t. $y_a, y_b, y_c \in \{0,1\}$

$$y_a + y_b \geq 1$$
$$y_a + y_c \geq 1$$
$$y_b + y_c \geq 1$$

$(y_a, y_b, y_c) = (1,1,0)$ feas.

$W = \{a, b\}$ vertex cover