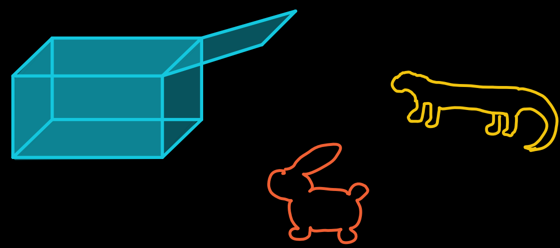
Math 407

Today: Examples of LP Models

Optimal Classifiers (§2.5 Separation of Points)

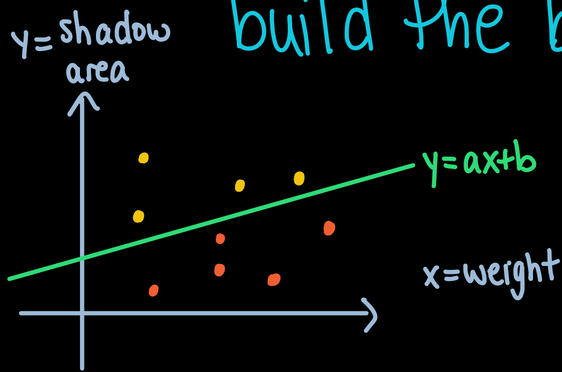
Computer controlled rabbit trap needs to trap rabbits that enter, but not trap weasels. It can tell

1) the weight and
2) the area of the shadow of the animal inside



animal $\longrightarrow$ pt in $\mathbb{R}^2$ (weight, shadow area)

Goal: Given data from k rabbits and ℓ weasels build the best linear classifier



Want $a, b \in \mathbb{R}^2$ s.t. all rabbit data satisfies $y < ax+b$ and all weasel data satisfies $y > ax+b$ (or vice versa)

Rabbit data (from k rabbits) : $r_1, \ldots, r_k \in \mathbb{R}^2$ $\quad r_i = (r_{i1}, r_{i2})$

Weasel data (from ℓ weasels) : $w_1, \ldots, w_\ell \in \mathbb{R}^2$ $\quad w_j = (w_{j1}, w_{j2})$

Each $r_i, w_j \in \mathbb{R}^2$ imposes an affine linear inequality on a, b

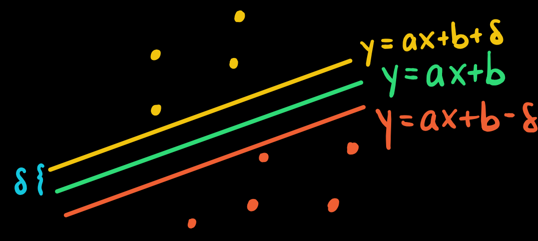
e.g. $r_{i2} < a r_{i1} + b$, $\quad w_{j2} > a w_{j1} + b$

Best separator?

max δ s.t.

$$r_{i2} < a r_{i1} + b - \delta \quad i = 1, .., k$$

$$w_{j2} > a w_{j1} + b + \delta \quad j = 1, .., \ell$$



← 3 variables, a, b, δ
$k+\ell$ inequalities

Linear separator exists $\Leftrightarrow$ opt val > 0

Use optimal separator $y = ax+b$ to program animal trap.

animal enters with $y < ax+b \to$ rabbit $\to$ trap

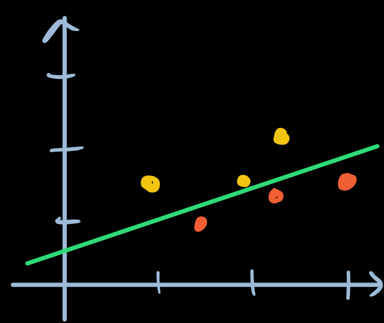
animal enters with $y > ax+b \to$ weasel $\to$ release

Ex: Rabbits: $(1.4, 1), (2.2, 1.5), (3, 1.7)$

Weasels: $(1, 1.5), (2, 2.2), (2.3, 2)$

Optimal sol: $(a, b, c) = (5/13, 23/26, 3/13)$

$\approx (.385, .885, .231)$



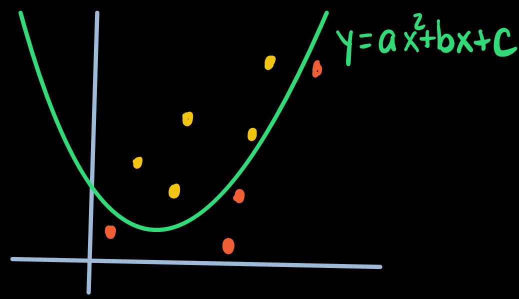
Nonlinear classifiers?

Pick any linear space of functions $\{a_1 f_1 + \ldots + a_m f_m : a_1, \ldots, a_m \in \mathbb{R}\}$ as potential classifiers where $f: \mathbb{R}^n \to \mathbb{R}$. Evaluation at a point $x \in \mathbb{R}^n$ is a linear function on parameters $a_1, .., a_n$!

Ex: Parabolic separators

$y < ax^2+bx+c$ for rabbits

$y > ax^2+bx+c$ for weasels

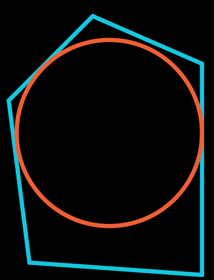


Points still give affine linear inequalities on (a,b,c):

e.g. $r_{i2} < a r_{i1}^2 + b r_{i1} + c, \quad w_{j2} > a w_{j1}^2 + b w_{j1} + c$

§2.6 Largest Ball in a Polyhedron

$P = \{ x \in \mathbb{R}^n \text{ s.t. } a_i^T x \le b_i \quad i=1,\dots,m \}$

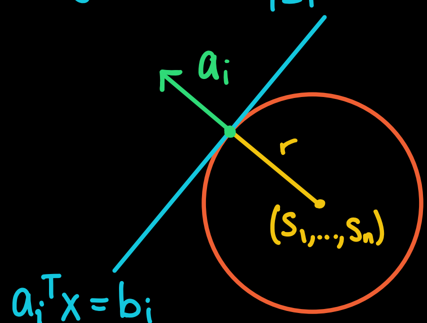


What is the largest radius of a ball contained in P?

Disk determined by radius r and center $s \in \mathbb{R}^n$

At first glance this is a very nonlinear problem!

When do all points in the ball $\{x \in \mathbb{R}^n : \sum_{i=1}^n (x_i - s_i)^2 \le r^2\}$ satisfy $a_i^T x \le b_i$?



Closest pt to s on the hyperplane $a_i^T x = b_i$?

Given by $s + \lambda a_i$ for some $\lambda \in \mathbb{R}$:

Needs to satisfy $a_i^T x = b_i$:

$$a_i^T(s+\lambda a_i) = b_i \implies a_i^T s + \lambda \|a_i\|^2 = b_i$$

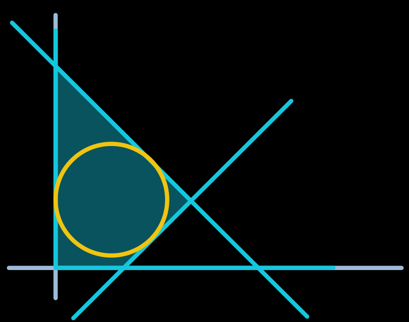
$$\implies \lambda = \frac{(b_i - a_i^T s)}{\|a_i\|^2}$$

Need $\lambda \geq 0$, $\|\lambda a_i\| = \lambda \|a_i\| = \dfrac{(b_i - a_i^T s)}{\|a_i\|} \geq r$

LP: $\max\ r$ s.t. $r \leq \frac{1}{\|a_i\|}(b_i - a_i^T s)$ $\quad i=1,\dots,m$

$n+1$ var, m ineq.

Ex: $P = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0,\ x_1+x_2 \leq 3,\ x_1 - x_2 \leq 1$



$a_1 = (-1,0)$ $\quad b_1 = 0 \longrightarrow r \leq s_1$

$a_2 = (0,-1)$ $\quad b_2 = 0 \longrightarrow r \leq s_2$

$a_3 = (1,1)$ $\quad b_3 = 1 \longrightarrow r \leq \frac{1}{\sqrt{2}}(1 - s_1 - s_2)$

$a_4 = (1,-1)$ $\quad b_3 = 1 \longrightarrow r \leq \frac{1}{\sqrt{2}}(1 - s_1 + s_2)$

Opt sol : $r = s_1 = \dfrac{2\sqrt{2}}{2+\sqrt{2}}$ $\qquad s_2 = 1$