Math 407

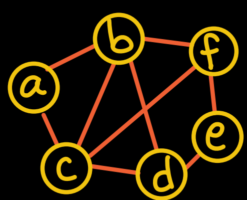
Today: An example of LP Duality in combinatorial optimization

Some benefits of LP Duality

- certification of optimality
- dual might be faster to solve than the primal
  (n variable, m equations $\longleftrightarrow$ n variable, n-m equations)
- can start dual simplex method at a non-feasible basis (if it is dual feasible)
- can reveal that seemingly different problems have the same solution.

Matchings and vertex covers in graphs

(more detail in Math 409)

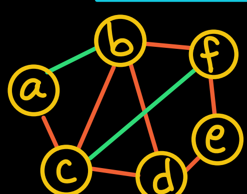


$G=(V,E)$ a graph, $V=$ "vertices"

$E=$ "edges" = collection of unordered pairs of vertices

ex: $V=\{$students in this class$\}$, $E=\{$pairs of friends$\}$

A <u>matching</u> is a set $M\subseteq E$ s.t. no edges in M share a vertex



ex: a way of pairing up (some) students so that every pair are friends.

Max Matching Problem: Given G, find $\max\{|M| : M \text{ is a matching}\}$

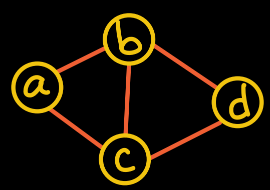
LP Relaxation: Given $G=(V,E)$, make variable x_e for all $e \in E$

"$x_e = 0$" for $e \notin M$, "$x_e = 1$" for $e \in M$

$$(P) \quad \max \sum_{e \in E} x_e \quad \text{s.t.} \quad x_e \geq 0 \text{ for all } e \in E$$
$$\sum_{e \ni v} x_e \leq 1 \text{ for all } v \in V$$

(with further constraint $x \in \mathbb{Z}^n$, this solves max matching!)

Ex:



$$(P) \quad \max \ x_{ab} + x_{ac} + x_{bc} + x_{bd} + x_{cd} \quad \text{s.t.}$$

$$\begin{aligned} x_{ab} + x_{ac} &\leq 1 \\ x_{ab} + x_{bc} + x_{bd} &\leq 1 \\ x_{ac} + x_{bc} + x_{cd} &\leq 1 \\ x_{bd} + x_{cd} &\leq 1 \end{aligned} \qquad \begin{pmatrix} x_{ab} \\ x_{ac} \\ x_{bc} \\ x_{bd} \\ x_{cd} \end{pmatrix} \geq 0$$

opt val: 2 an optimal solution: $x_{ab} = x_{cd} = 1$, $x_{ac} = x_{bc} = x_{bd} = 0$

This LP is $\max \mathbb{1}_E^T x$ s.t. $Ax \leq \mathbb{1}_V$, $x \geq 0$

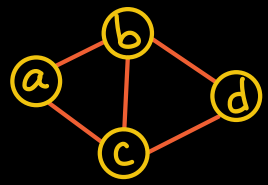
where A is a $|V| \times |E|$ matrix with $A_{ve} = \begin{cases} 1 & \text{if } v \in e \\ 0 & \text{o.w.} \end{cases}$

One formulation of the dual is

$$(D) \quad \min \mathbb{1}_V^T y \quad \text{s.t.} \quad y \geq 0, \ A^T y \geq \mathbb{1}_E \qquad (y \in \mathbb{R}^V)$$

That is
$$\min \sum_{v \in V} y_v \quad \text{s.t.} \quad y_v \geq 0 \text{ for all } v \in V$$
$$\sum_{v \in e} y_v \geq 1 \text{ for all } e \in E$$

What problem does this solve? After adding $y \in \mathbb{Z}^V$?

Ex: 

$\min\ y_a+y_b+y_c+y_d$ s.t.

$y_a+y_b \ge 1$ $\quad y_b+y_d \ge 1$

$y_a+y_c \ge 1$ $\quad y_c+y_d \ge 1$

$y_b+y_c \ge 1$

$\begin{pmatrix} y_a \\ y_b \\ y_c \\ y_d \end{pmatrix} \ge 0$

Opt val: 2 an optimal solution: $y_a=y_d=0,\ y_b=y_c=1$

This also gives certificate of optimality for (P):

$$1(x_{ab}+x_{bc}+x_{bd} \le 1) + 1(x_{ac}+x_{bc}+x_{cd} \le 1) + (-x_{bc} \le 0)$$

$\Rightarrow$ $x_{ab}+x_{ac}+x_{bc}+x_{bd}+x_{cd} \le 2$ on (P)

Similarly x^* gives upper bound on (D):

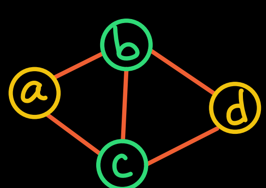
$$(y_a+y_b \ge 1) + (y_c+y_d \ge 1) \Rightarrow y_a+y_b+y_c+y_d \ge 2$$

With the added constraint $y \in \mathbb{Z}^V$, this solves:

Min Vertex Cover Problem:

Find $\min\{|W| : W \subseteq V$ has at least one vertex in every edge$\}$

"$y_v = 1$" for $v \in W$ "$y_v = 0$" for $v \notin W$

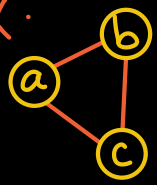


$\leftarrow W=\{b,c\}$ is a vertex cover with $|W|=2$

Application: Find smallest # guards s.t. every corridor is guarded

Warning: Sometimes the LP relaxations don't solve the combinatorial problem.

Ex:



(P) max $x_{ab}+x_{ac}+x_{bc}$ s.t. $x_{ab}+x_{ac}\le 1$, $x_{ab}+x_{bc}\le 1$, $x_{ac}+x_{bc}\le 1$, $\begin{pmatrix} x_{ab} \\ x_{ac} \\ x_{bc} \end{pmatrix} \ge 0$

opt. val $= 3/2 > 1 =$ max matching

opt sol: $(1/2, 1/2, 1/2)$

(D) min $y_a+y_b+y_c$ s.t. $y_a+y_b\ge 1$, $y_a+y_c\ge 1$, $y_b+y_c\ge 1$, $\begin{pmatrix} y_a \\ y_b \\ y_c \end{pmatrix} \ge 1$

opt val $= 3/2 < 2 =$ min vertex cover

opt sol $= (1/2, 1/2, 1/2)$