Math 407

Today: Dual Simplex Method

Start with primal LP is equational form:

$$(P) \quad \max\ c^T x \ \text{ s.t. } Ax = b,\ x \geq 0.$$

$A \in \mathbb{R}^{m \times n} \Rightarrow$
m equations
n variables

By Hwk 5, Prob 3, the dual LP is given by

$$(D) \quad \min\ b^T y \ \text{ s.t. } A^T y - s = c,\ s \geq 0$$

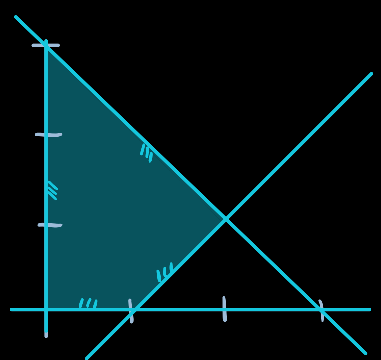
By Hwk 6, Prob 3, we can reformulate this as an LP

$$(D') \quad \min\ d^T s + \underbrace{d^T c}_{\text{constant}} \ \text{ s.t. } Ms = -Mc,\ s \geq 0$$

n-m equations
n variables

Ex: (P) $\max\ -2x_1 - x_2$ s.t.

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - x_2 + x_4 &= 1 \end{aligned} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geq 0$$



$Ax = b$ for $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$ $\quad b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

By Hwk 6, Prob 3, take $M = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$, $d = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \end{pmatrix}$

(D) $\min\ 3s_3 + s_4$ s.t.

$$\begin{aligned} s_1 - s_3 - s_4 &= 2 \\ s_2 - s_3 + s_4 &= 1 \end{aligned} \qquad \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} \geq 0$$

You will show that if $A = (-Q \ \ I_m)$, then we can take $M = (I_{n-m} \ \ Q^T)$.

$\{n-m+1, \ldots, n\}$ is a basis for (P)

$\{1, \ldots, n-m\}$ is a basis for (D)

Corollary: $B \subseteq \{1,\dots,n\}$ with $|B|=m$ is a basis for (P) if and only if $N=\{1,\dots,n\}\setminus B$ is a basis for (D).

$T(B)$ for primal $\Leftrightarrow$ $T(N)$ for dual

$$x_B = p + Q x_N \qquad s_N = -r - Q^T s_B$$
$$z = z_0 + r^T x_N \qquad z = z_0 + p^T s_B$$

Ex (cont') $B=\{3,4\}$, $N=\{1,2\}$

$T(B)$ for primal:
$$x_3 = 3 - x_1 - x_2$$
$$x_4 = 1 - x_1 + x_2$$
$$z = 0 - 2x_1 - x_2$$

$T(N)$ for dual:
$$s_1 = 2 + s_3 + s_4$$
$$s_2 = 1 + s_3 - s_4$$
$$z = 0 + 3s_3 + s_4$$

We can solve (D) working only with tableau for (P).

The Dual Simplex method

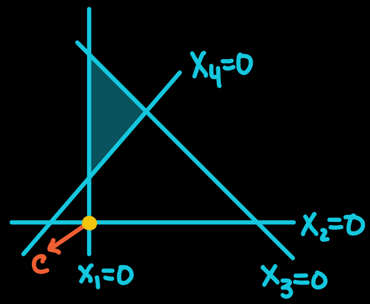
(see Glossary of MG book)

(P) $\max c^T x$ s.t. $Ax=b$, $x \geq 0$

We call a basis $B \subseteq \{1,\dots,n\}$ <u>dual feasible</u> if $r \leq 0$ in the simplex tableau $T(B)$.

Note: If B is both feasible ($p \geq 0$) and dual feasible ($r \leq 0$) then it corresponds to the optimal solution!

Ex: max $-2x_1 - x_2$ s.t. $x_1 + x_2 + x_3 = 3$, $x_1 - x_2 + x_4 = -1$, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geq 0$



$T(\{3,4\})$
$$x_3 = 3 - x_1 - x_2$$
$$x_4 = -1 - x_1 + x_2$$
$$z = 0 - 2x_1 - x_2$$

$\Rightarrow B = \{3,4\}$ is dual feasible ($r \leq 0$) but not feasible

Starting from a dual feasible basis we can pivot in the dual LP

<u>Dual Pivot Step</u>: Move from one dual feasible basis $B \subseteq \{1, \ldots, n\}$ to another $B' = B \cup \{i\} \setminus \{j\}$

x_i = "the entering variable" ← previously constrained to be zero by $x_k = 0 \; \forall k \notin B$
x_j = "the leaving variable" ← newly constrained to be zero by $x_k = 0 \; \forall k \notin B$

<u>Start</u> with a dual feasible basis B, $T(B)$: $x_B = p + Qx_N$, $z = z_0 + r^T x_N$ $(r \leq 0)$

1) If $p \geq 0$, STOP. Opt. val. $= z_0$, Opt. sol: $(x_B, x_N) = (p, 0)$

Otherwise, choose leaving variable x_j with $p_j < 0$.

Idea: Add some nonneg. scalar multiple of j^{th} eq.
$x_j = p_j + q_j^T x_N \Leftrightarrow 0 = p_j - x_j + q_j^T x_N$ to $z = z_0 + r^T x_N$
while maintaining coeff of $x_N \leq 0$.

2) If j^{th} row q_j^T of Q is ≤ 0 then (P) is infeasible.

$$\left(x_j = p_j + q_j^T x_N \Rightarrow \underbrace{x_j - q_j^T x_N}_{\text{all coeff are } \ge 0} = \underbrace{p_j}_{<0}\right)$$

Otherwise, find max value λ^* of λ s.t. $r + \lambda q_j \le 0$.
Choose entering variable x_i s.t. $(r + \lambda^* q_j)_i = 0$.
New basis: $B' = B \cup \{i\} \setminus \{j\}$

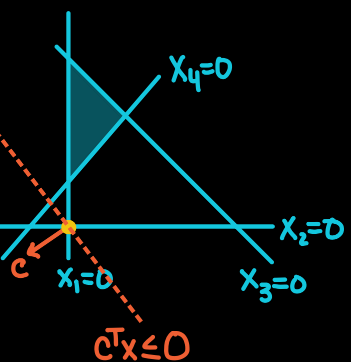
This is really the usual simplex method running on the dual linear program!

In Ex: $T(\{3,4\})$

$$x_3 = 3 - x_1 - x_2$$
$$x_4 = \boxed{-1} - x_1 + x_2$$
$$z = 0 - 2x_1 - x_2$$

λ

$p_4 < 0 \Rightarrow$ x_4 leaves



$$\lambda x_4 + z = -\lambda + \underline{(-\lambda-2)}x_1 + \underline{(\lambda-1)}x_2$$

$$-\lambda - 2 \le 0 \qquad \lambda - 1 \le 0$$

Taking λ as large as possible while keeping coeff ≤ 0
$\Longrightarrow$ $\lambda = 1$, makes (coeff of x_2) $= 0$ $\Rightarrow$ x_2 enters

$T(\{2,3\})$

$$x_2 = 1 + x_1 + x_4$$
$$x_3 = 2 - 2x_1 - x_4$$
$$z = -1 - 3x_1 - x_4$$

feasible and dual feasible
$\Rightarrow$ optimal!