Math 407

Today: Proofs of Strong Duality

Thm (Strong Duality, version 1)

If (P) is feasible and bounded, then there exists x^* feasible for (P) and y^* feasible for (D) with $c^Tx^* = b^Ty^*$.

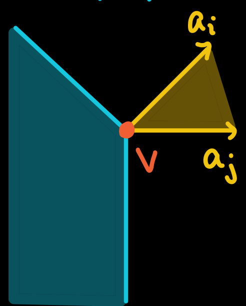
$\Rightarrow$ (P) and (D) have the same optimal values.

namely $c^Tx^* = b^Ty^*$

(Proof by Farkas Lemma/Convex Geometry, sketch §6.4)

(P) $\max c^Tx$ s.t. $Ax \le b$ (i.e. $a_i^Tx \le b_i$ $i = 1, \dots, m$)

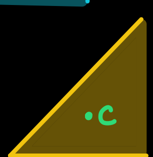
(D) $\min b^Ty$ s.t. $A^Ty = c$, $y \ge 0$



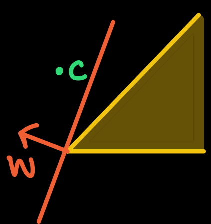
Suppose v is feasible for (P)

Let $I = \{i : a_i^Tv = b_i\}$. ← ineq hold with "=" at v

Either (1) c belongs to the convex cone



$$C_v = \left\{ \sum_{i \in I} \lambda_i a_i : \lambda_i \ge 0 \ \forall i \right\}$$



or (2) there exists $w \in \mathbb{R}^n$ s.t. $w^Ta_i \le 0$ for all $i \in I$ and $w^Tc > 0$. (Why does w exist? See §6.4)

If (1), $c = \sum_{i \in I} \lambda_i a_i$ with $\lambda_i \ge 0$. Take $y \in \mathbb{R}^m$ with $y_i = \lambda_i$ for $i \in I$, $y_j = 0$ for $j \notin I$. Then y is feasible for (D) and $c^Tv = b^Ty$ by complimentary slackness $(y_k(b_k - a_k^Tv) = 0 \ \forall k)$

$\Rightarrow$ v is optimal for (P), y is optimal for (D)

If (2), then $v+\varepsilon w$ is feasible for some $\varepsilon > 0$.

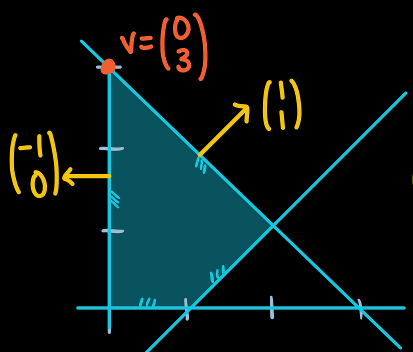
Check: $(i \in I)$ $a_i^T(v+\varepsilon w) = a_i^T v + \varepsilon a_i^T w = b_i + \varepsilon \underbrace{w^T a_i}_{\leq 0} \leq b_i$

$(j \notin I)$ $a_j^T(v+\varepsilon w) = \underbrace{a_j^T v}_{< b_j} + \varepsilon a_j^T w \leq b_j$ for small enough $\varepsilon > 0$.

(namely $\varepsilon < (b_j - a_j^T v)/|a_j^T w|$)

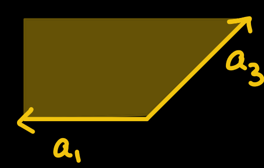
Since $c^T w > 0$, $c^T(v+\varepsilon w) = c^T v + \varepsilon c^T w > c^T v$ $\Rightarrow$ v is not optimal

Ex: (P) max $c_1 x_1 + c_2 x_2$ s.t. $\underset{(1)}{-x_1 \leq 0}$, $\underset{(2)}{-x_2 \leq 0}$, $\underset{(3)}{x_1 + x_2 \leq 3}$, $\underset{(4)}{x_1 - x_2 \leq 1}$

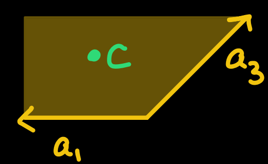


$v = \begin{pmatrix}0\\3\end{pmatrix} \Rightarrow I = \{1,3\}$

$C_v = \left\{ \lambda_1 \begin{pmatrix}-1\\0\end{pmatrix} + \lambda_3 \begin{pmatrix}1\\1\end{pmatrix} : \lambda_1, \lambda_3 \geq 0 \right\}$

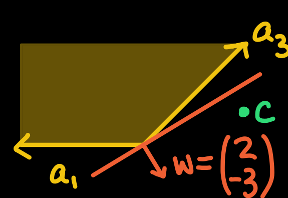


ex 1: $c = \begin{pmatrix}-1\\1\end{pmatrix} = 2\begin{pmatrix}-1\\0\end{pmatrix} + 1\begin{pmatrix}1\\1\end{pmatrix} \in C_v$



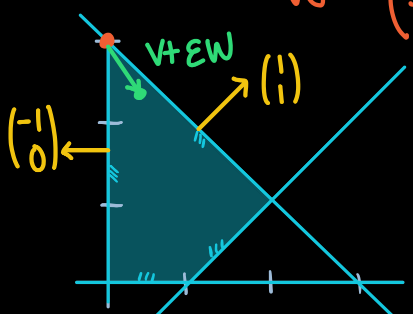
$\Rightarrow$ v optimal, $y^T = (2,0,1,0)$ opt. for (D).

ex 2: $c = \begin{pmatrix}2\\1\end{pmatrix} \notin C_v$



$w = \begin{pmatrix}2\\-3\end{pmatrix}$ has $w^T u \leq 0$ for all $u \in C_v$

$w^T c = 1 > 0$



$\Rightarrow v+\varepsilon w = \begin{pmatrix}2\varepsilon\\3-3\varepsilon\end{pmatrix}$ feasible for some $\varepsilon > 0$

$c^T(v+\varepsilon w) = c^T v + \varepsilon c^T w = 3 + \varepsilon > 3 = c^T v$.

(Proof by termination of the Simplex Method)

(P) $\max c^T x$ s.t. $Ax = b,\ x \geq 0$.

(D) $\min b^T y$ s.t. $A^T y - s = c,\ s \geq 0$

If (P) is feasible and bounded, then some feasible basis B has simplex tableau:

Same sol as $Ax=b$, $z=c^Tx$ $\left\{ \begin{aligned} x_B &= p + Q x_N \\ z &= z_0 + r^T x_N \end{aligned} \right.$ with $p \geq 0,\ r \leq 0$.

Opt sol for primal: $x^* = (x_B^*, x_N^*) = (p, 0)$

Opt sol for dual: $s^* = (s_B^*, s_N^*) = (0, -r)$

$Ax=b$, $z = c^T x$ has same sol. as $Ax = b$, $z = z_0 - (s^*)^T x$

$\hookrightarrow \begin{pmatrix} A & 0 \\ -c^T & 1 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} \xrightarrow{\text{row op.}} \begin{pmatrix} A & 0 \\ (s^*)^T & 1 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} b \\ z_0 \end{pmatrix}$

$\Rightarrow -c^T - (s^*)^T \in \text{rowspan}(A)$

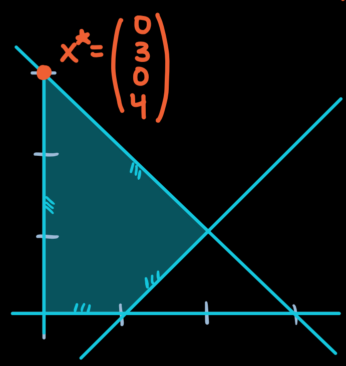
$\Rightarrow c^T + (s^*)^T = (y^*)^T A$ for some $y^* \in \mathbb{R}^m$

$\Rightarrow$ If $r \leq 0$, then (y^*, s^*) dual feasible $(A^T y^* - s^* = c,\ s^* \geq 0)$ and optimal by Complimentary slackness ($x_i^* s_i^* = 0$ for all i)

Hwk Prob 3!

Ex: (P) max $-x_1+x_2$ s.t. $x_1+x_2+x_3=3$ $\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} \geq 0$

$x_1-x_2+x_4=1$



Opt sol: $x^* = (0,3,0,4)$ Basis: $B=\{2,4\}$

$T(\{3,4\})$
$$\begin{aligned} x_3 &= 3 - x_1 - \underline{x_2} \\ x_4 &= 1 - x_1 + x_2 \\ z &= 0 - x_1 + \underline{\underline{x_2}} \end{aligned}$$

x_2 enters
x_3 leaves

$T(\{2,4\})$
$$\begin{aligned} x_2 &= 3 - x_1 - x_3 \\ x_4 &= 4 - 2x_1 - x_3 \\ z &= 3 \underline{\underline{-2}}x_1 \underline{\underline{-}}x_3 \end{aligned}$$

Opt sol for (D) has $s^* = (2,0,1,0)$

$$\left(\begin{array}{cc|c} A & 0 & b \\ -c^T & 1 & 0 \end{array}\right): \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 3 \\ 1 & -1 & 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 \end{array}\right)$$

add row 1 to row 2

add row 1 to row 3

$$\left(\begin{array}{cc|c} A & 0 & b \\ (s^*)^T & 1 & z_0 \end{array}\right): \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 3 \\ 2 & 0 & 1 & 1 & 0 & 4 \\ 2 & 0 & 1 & 0 & 1 & 3 \end{array}\right)$$

$$c^T + (s^*)^T = (-1,1,0,0) + (2,0,1,0) = (1,1,1,0) = \underline{\underline{(1,0)}}\,A$$

$$y^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$