Math 407

Today: Complimentary Slackness

(P) $\max c^T x$ s.t. $Ax \le b$

(D) $\min b^T y$ s.t. $A^T y = c$, $y \ge 0$

Benefits of LP Duality

1) In practice, complexity of solving (P) depends on m (# equations). If $m > n-m$, it may be much faster to solve (D).

2) Gives concise certificate for optimality, (feasibly x^*, y^* s.t. $c^T x^* = b^T y^*$)

Another way to check optimality:

Complimentary Slackness

Useful fact: If $\lambda_1, \dots, \lambda_n \ge 0$ and $\lambda_1 + \dots + \lambda_n = 0$, then $\lambda_1 = 0, \dots, \lambda_n = 0$

If x is feasible for (P), y feasible for (D) then

$$0 \le b_i - a_i^T x, \quad y_i \ge 0, \text{ and } \sum_{i=1}^m y_i a_i^T x = c^T x$$

$$\Rightarrow b^T y - c^T x = \sum_{i=1}^m \underbrace{y_i (b_i - a_i^T x)}_{\text{each term } \ge 0}$$

Then $c^T x = b^T y \iff y_i(b_i - a_i^T x) = 0$ for all $i = 1, \dots, m$

Thm (Complimentary Slackness) For the LPs

(P) $\max c^T x$ s.t. $Ax \leq b$

(D) $\min b^T y$ s.t. $A^T y = c,\ y \geq 0,$

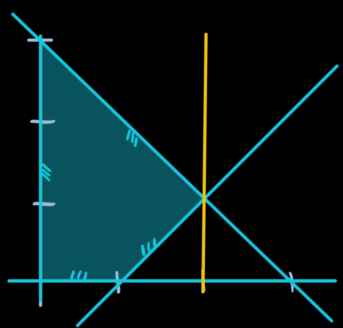
if x is feasible for (P) and y is feasible for (D), then x and y are optimal (for (P) and (D) resp.)

$\iff c^T x = b^T y$

$\iff y_i(b_i - a_i^T x) = 0$ for all $i = 1, \ldots, m$.

Ex 1: (P) $\max x_1$ s.t. $\underbrace{-x_1 \leq 0}_{a_1^T x \leq b_1},\ \underbrace{-x_2 \leq 0}_{a_2^T x \leq b_2},\ \underbrace{x_1 + x_2 \leq 3}_{a_3^T x \leq b_3},\ \underbrace{x_1 - x_2 \leq 1}_{a_4^T x \leq b_4}$

(D) $\min 3y_3 + y_4$ s.t. $-y_1 + y_3 + y_4 = 1$
$\qquad -y_2 + y_3 - y_4 = 0,\quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \geq 0$



Opt sol for (P): $x^* = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Which ineq. $a_i^T x \leq b_i$ do not hold with "=" at x^*?

$$\left.\begin{array}{l} a_1^T x^* < b_1 \\ a_2^T x^* < b_2 \\ a_3^T x^* = b_3 \\ a_4^T x^* = b_4 \end{array}\right] \rightarrow$$

y is optimal for (D)

$\iff y$ is feasible for (D) $\left(\sum_{i=1}^m y_i a_i^T = c_i,\ y \geq 0\right)$ and $y_1 = y_2 = 0$

$\iff y_1 = y_2 = 0,\quad y_3 + y_4 = 1,\quad y_3 \geq 0$
$\qquad\qquad\qquad\quad y_3 - y_4 = 0,\quad y_4 \geq 0$

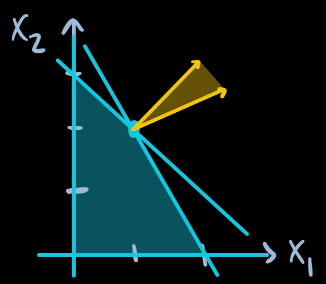
$\iff y_1 = y_2 = 0,\quad y_3 = y_4 = 1/2 \leftarrow$ unique sol to these linear equations!

Why not $y^T=(1,0,1,1)$?

$$b^Ty-c^Tx^* = \underbrace{y_1}_{>0}\underbrace{(b_1-a_1^Tx^*)}_{>0}+\underbrace{y_2}_{=0}(b_2-a_2^Tx^*)+y_3^*\underbrace{(b_3-a_3^Tx^*)}_{=0}+y_4\underbrace{(b_4-a_4^Tx^*)}_{=0}$$

$$>0$$

Geometry of Complimentary Slackness



If x^* is an optimal solution of (P), the best upper bound only uses inequalities that are tight at x^*.

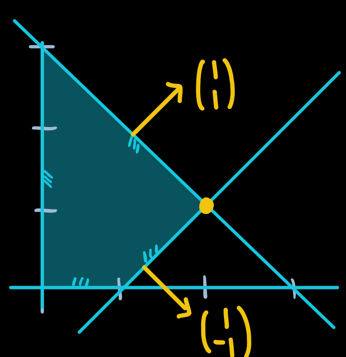
Cor: If v is a vertex of $P=\{x\in\mathbb{R}^n : Ax\leq b\}$ then the maximum of c^Tx over P is attained at v if and only if c is a nonnegative combination of the vectors a_i with $a_i^Tv=b_i$.

(Proof) x^* acheives max c^Tx s.t. $a_i^Tx\leq b_i$ for $i=1,\dots,m$

$$\iff \exists\, y^*\geq 0 \text{ s.t. } \sum_{i=1}^m y_i^* a_i^T = c^T$$

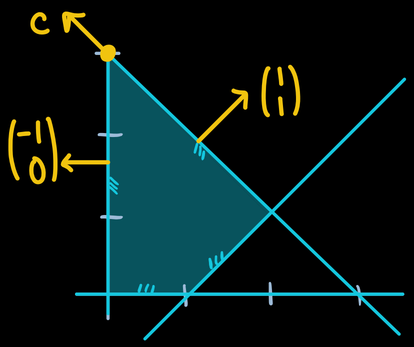
with $y_i^*=0$ for i s.t. $a_i^Tx<b_i$.

Ex: (P) max $c_1x_1+c_2x_2$ s.t. $\underset{a_1^Tx\leq b_1}{-x_1\leq 0}$, $\underset{a_2^Tx\leq b_2}{-x_2\leq 0}$, $\underset{a_3^Tx\leq b_3}{x_1+x_2\leq 3}$, $\underset{a_4^Tx\leq b_4}{x_1-x_2\leq 1}$



Comp. Slackness: max acheived at $x=(2,1)$

$$\iff c=y_3\begin{pmatrix}1\\1\end{pmatrix}+y_4\begin{pmatrix}1\\-1\end{pmatrix} \text{ for some } y_3\geq 0,\ y_4\geq 0$$



Q: For which c is the max acheived at $v = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$?

$a_1^T v = b_1$
$a_2^T v < b_2$
$a_3^T v = b_3$
$a_4^T v < b_4$

A: $c = y_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + y_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

for some $y_1 \geq 0,\ y_3 \geq 0$

Eg. For $c = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, max $c^T x$ acheived at $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

$\Rightarrow c = y_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + y_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\left[\begin{array}{l} -1 = -y_1 + y_3 \\ 1 = 0 + y_3 \end{array}\right]$ unique sol: $y_1 = 2,\ y_3 = 1$

$\Rightarrow$ optimal sol for (D): $y^* = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$