Math 407

Today: Duality cont' (Other formulations and strong duality)

From last time....

given a (primal) linear program

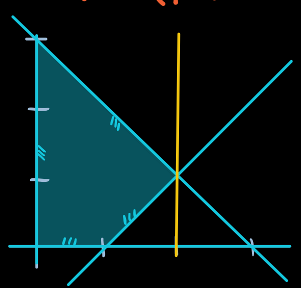
$$(P)\quad \max\ c^T x \quad \text{s.t. } Ax \le b.$$

the dual linear program

$$(D)\quad \min\ b^T y \quad \text{s.t. } A^T y = c,\ y \ge 0$$

searches for the best possible upper bound on $c^T x$ of the form $\sum_{i=1}^{m} y_i \,(a_i^T x \le b_i)$.

Ex 1: (P) $\max x_1$ s.t. $\underbrace{\overset{y_1}{-x_1 \le 0},\ \overset{y_2}{-x_2 \le 0},\ \overset{y_3}{x_1 + x_2 \le 3},\ \overset{y_4}{x_1 - x_2 \le 1}}$



$Ax \le b$ for $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad b = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \end{pmatrix}$

(D) $\min\ 3y_3 + y_4$ s.t. $\underbrace{\begin{aligned} -y_1 + y_3 + y_4 &= 1 \\ -y_2 + y_3 - y_4 &= 0, \end{aligned}}$ $\quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \ge 0$

$A^T y = c \longrightarrow$

e.g. all feas. for (D):

$y = (1,0,1,1) \Rightarrow x_1 \le 4$

$y = (7,2,5,3) \Rightarrow x_1 \le 18$

$y = (0,0,1/2,1/2) \Rightarrow x_1 \le 2$

all valid upper bounds on (P)

Other formulations?

(P) $\max c^T x$ s.t. $Ax \le b,\ x \ge 0$

(D) $\min b^T y$ s.t. $A^T y \ge c,\ y \ge 0$

Why? $Ax \le b,\ x \ge 0 \iff \begin{pmatrix} A \\ -I \end{pmatrix} x \le \begin{pmatrix} b \\ 0 \end{pmatrix}$

Dual: $\min b^T y$ s.t. $\underbrace{\begin{pmatrix} A^T & -I \end{pmatrix} \begin{pmatrix} y \\ s \end{pmatrix} = c,\ \begin{pmatrix} y \\ s \end{pmatrix} \ge 0}_{\text{equiv. to } A^T y \ge c,\ y \ge 0}$

In Ex 1, dual (D) equivalent to

$$\min 3y_3 + y_4 \text{ s.t. } \begin{array}{l} y_3 + y_4 \ge 1 \\ y_3 - y_4 \ge 0, \end{array} \quad \begin{pmatrix} y_3 \\ y_4 \end{pmatrix} \ge 0$$

(P) $\max c^T x$ s.t. $Ax = b,\ x \ge 0$

(D) $\min b^T y$ s.t. $A^T y - s = c,\ s \ge 0$

or equivalently:

(D′) $\min b^T y$ s.t. $A^T y \ge c$

See Problem 3 from Hwk!

Remark: The dual of the dual of an LP is the primal!
Details in upcoming homework.

Thm (Strong Duality, version 1)

If (P) is feasible and bounded, then there exists x^* feasible for (P) and y^* feasible for (D) with $c^T x^* = b^T y^*$.

$\Rightarrow$ (P) and (D) have the same optimal values.

namely $c^T x^* = b^T y^*$

(Proof later!)

Thm (Strong Duality, version 2)

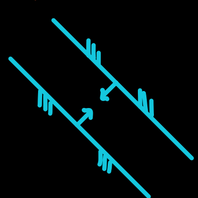
Exactly one of the following hold:

1) Both (P) and (D) are infeasible
2) (P) is unbounded and (D) is infeasible
3) (P) is infeasible and (D) is unbounded
4) Both (P) and (D) are feasible and have the same optimal value.

Ex (1): (P) max x_1 s.t. $x_1 + x_2 \geq 1$, $\underset{y_2}{x_1 + x_2 \leq 0}$

$\Updownarrow$

$\underset{y_1}{-x_1 - x_2 \leq -1}$



(D) min $-y_1$ s.t. $y_1 \geq 0$, $y_2 \geq 0$,

$-y_1 + y_2 = 1$ (coeff of x_1)

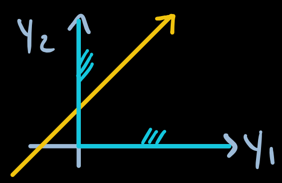
$-y_1 + y_2 = 0$ (coeff of x_2)

Both (P), (D) infeasible!

Ex (3) (P) max x_1+x_2 s.t. $x_1+x_2 \geq 1$, $x_1+x_2 \leq 0$

$-x_1-x_2 \leq -1$

(D) min $-y_1$ s.t. $y_1 \geq 0$, $y_2 \geq 0$,



$-y_1+y_2 = 1$ (coeff of x_1)

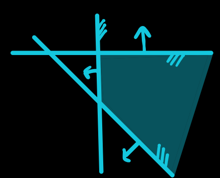
$-y_1+y_2 = 1$ (coeff of x_2)

(D) unbounded (from below!) $(y_1,y_2) = (t, 1+t)$ feas. $\forall t \geq 0$

obj. function $-y_1 = -t \to -\infty$ as $t \to \infty$

Ex (2): (P) max x_1 s.t. $x_1+x_2 \geq 1$, $x_1 \geq 2$, $x_2 \leq 0$,

$-x_1-x_2 \leq -1$, $-x_1 \leq -2$, $x_2 \leq 0$



(P) unbounded, $(2+t,-1)$ feasible for all $t \geq 0$

(D) min $-y_1-2y_2$ s.t. $(y_1,y_2,y_3) \geq 0$, $-y_1-y_2 = 1$

$-y_1+y_3 = 0$

$y_1 \geq 0$, $y_2 \geq 0$, $-y_1-y_2=1$ impossible

$\Rightarrow$ (D) infeasible! $(1,0)$ not a nonneg. comb. of $(-1,-1)$, $(-1,0)$, $(0,1)$