Math 407

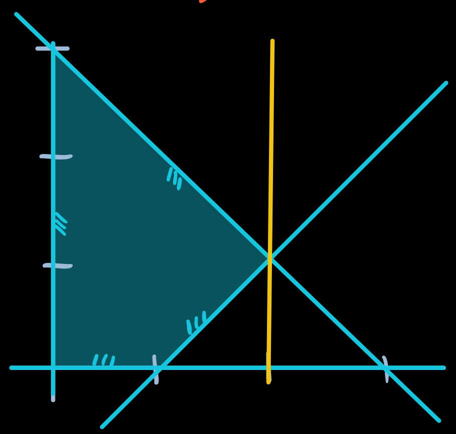
Today: Duality

Duality in Linear Programming

Idea: to a linear program (P) (for "primal")
we associate a dual linear program (D) that aims
to find the best possible upper bound on (P)

Ex 1: max x_1 s.t.

$x_1 \geq 0$, $x_2 \geq 0$, $x_1 + x_2 \leq 3$, $x_1 - x_2 \leq 1$



Optimal val = 2

Can we see $x_1 \leq 2$ from inequalities?

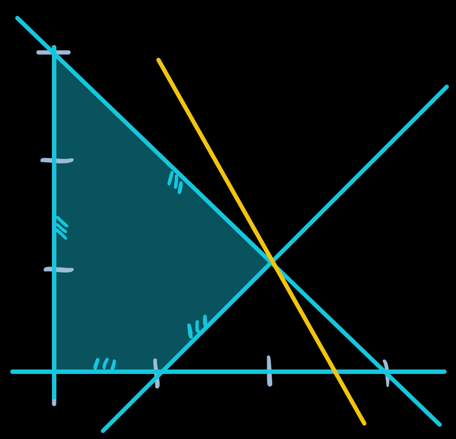
Yes: $x_1 + x_2 \leq 3 \xRightarrow[\times 1/2]{} \frac{1}{2}x_1 + \frac{1}{2}x_2 \leq \frac{3}{2}$

$x_1 - x_2 \leq 1 \xRightarrow[\times 1/2]{} \frac{1}{2}x_1 - \frac{1}{2}x_2 \leq \frac{1}{2}$

sum: $x_1 \leq 2$

Ex 2: max $2x_1 + x_2$

$x_1 \geq 0$, $x_2 \geq 0$, $x_1 + x_2 \leq 3$, $x_1 - x_2 \leq 1$



Optimal val = 5

Can we see $2x_1 + x_2 \leq 5$ from inequalities?

Yes: $x_1 + x_2 \leq 3 \xRightarrow[\times 3/2]{} \frac{3}{2}x_1 + \frac{3}{2}x_2 \leq \frac{9}{2}$

$x_1 - x_2 \leq 1 \xRightarrow[\times 1/2]{} \frac{1}{2}x_1 - \frac{1}{2}x_2 \leq \frac{1}{2}$

sum: $2x_1 + x_2 \leq 5$

What inequalities $c^T x \le c_0$ hold on $P = \{x \in \mathbb{R}^n : Ax \le b\}$?

Know inequalities $a_j^T x \le b_j$ hold on P

$\Rightarrow$ for any $y_1 \ge 0, \dots, y_m \ge 0$, $\left.\begin{array}{c} y_1 a_1^T x \le y_1 b_1 \\ \vdots \\ y_m a_m^T x \le y_m b_m \end{array}\right\}$ all hold on P

$$\Rightarrow \quad y^T A x = \left(\sum_{j=1}^m y_j a_j^T\right) x \le \left(\sum_{j=1}^m y_j b_j\right) = y^T b$$

when $y_1 \ge 0, \dots, y_m \ge 0$, this inequality "obviously" holds on P

If $y^T A$ equals cost vector c^T, this gives an upper bound (of $y^T b$) on

$$(P) \quad \max\ c^T x \quad \text{s.t.} \quad Ax \le b.$$

In Ex 1, taking $y^T = (1/2, 1/2, 1, 1/2)$ gives

$$\begin{array}{rl} 1/2 \times (-x_1 & \le 0) \\ 1/2 \times (\quad -x_2 & \le 0) \\ 1 \times (x_1 + x_2 & \le 3) \\ 1/2 \times (x_1 - x_2 & \le 1) \\ \hline x_1 & \le 7/2 \end{array}$$

This shows that $x_1 \le 7/2$ is a valid inequality on $\{x \in \mathbb{R}^2 : x_1 \ge 0, x_2 \ge 0, x_1 + x_2 \le 3, x_1 - x_2 \le 1\}$.

We saw above that an even better bound of $x_1 \le 2$ holds.

The <u>dual linear program</u> is

(D) $\qquad \min b^T y$ s.t. $A^T y = c$ and $y \geq 0$.

This looks for the best "obvious" upper bound on (P).

<u>Prop (Weak Duality)</u> If x is feasible for (P) and y is feasible for (D), then

$$c^T x \leq b^T y.$$

(Proof) x feasible for (P) $\iff Ax \leq b$

y feasible for (D) $\iff A^T y = c,\ y \geq 0$

If both hold,

$$c^T x = (A^T y)^T x = y^T A x \underset{\substack{\uparrow \\ \text{uses } y \geq 0}}{\leq} y^T b = b^T y$$

Consequences:

1) (P) unbounded (from above) $\Rightarrow$ (D) infeasible.
2) (D) unbounded (from below) $\Rightarrow$ (P) infeasible.
3) If x^* is feasible for (P), y^* is feasible for (D), and $c^T x^* = b^T y^*$ then both are optimal for (P), (D) resp.

Ex 1: (P) $\max x_1$ s.t. $\underbrace{\overset{y_1}{-x_1 \le 0}, \ \overset{y_2}{-x_2 \le 0}, \ \overset{y_3}{x_1 + x_2 \le 3}, \ \overset{y_4}{x_1 - x_2 \le 1}}$

$Ax \le b$ for $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad b = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \end{pmatrix}$

(D) $\min 3y_3 + y_4$ s.t. $\underbrace{\begin{aligned} -y_1 + y_3 + y_4 &= 1 \\ -y_2 + y_3 - y_4 &= 0, \end{aligned}}$ $\quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \ge 0$

$A^T y = c$

$x^* = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ feasible for (P)

$y^* = \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{pmatrix}$ feasible for (D)

$x_1^* = 2 = 3y_3^* + y_4^*$

$\Rightarrow$ x opt. sol. for (P) and y opt sol. for (D)