Math 407

Midterm – in class Wed. Nov. 1

Today: Review

Most general def of a linear program:

maximize/minimize a linear function subject to affine-linear equations and inequalities.

Can always reformulate as

(1) $\max c^T x$ s.t. $Ax \le b$

(2) $\max c^T x$ s.t. $Ax = b$, $x \ge 0$ (equational form)

Feasible region P is convex

($v, w \in P$, $\lambda \in [0,1] \Rightarrow \lambda v + (1-\lambda) w \in P$)

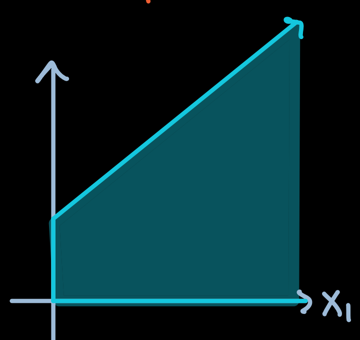
Important terminology:

feasible/infeasible, LP bounded/unbounded

feasible region bounded/unbounded

optimal value/optimal solution.

Ex (Practice Midterm) $\max c^T x$ s.t. $x_1 \ge 0, x_2 \ge 0, x_2 - x_1 \le 1$



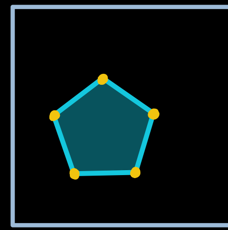
$c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $c^T x = x_2$ LP feasible, unbounded

$(x_1, 1+x_1)$ feas. for all $x_1 \ge 0$, $x_2 = 1 + x_1 \to \infty$ as $x_1 \to \infty$

$c = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $c^T x = -x_2$, LP feasible, bounded

$-x_2 \le 0$ on feas. region

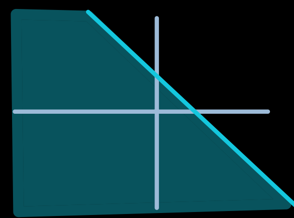
Thm: If P is bounded and nonempty P is the convex hull of its vertices



$\Rightarrow$ optimum acheived by a vertex

Thm: The optimal value of a bounded, feasible LP of the form $\max\{c^T x : Ax=b, x\geq 0\}$ is acheived at a vertex.

$\uparrow$ necessary



Non-ex: $\max\ x_1+x_2$ s.t. $x_1+x_2\leq 1$

LP is bounded and feasible

but opt. value not attained at a vertex (there are no vertices!)

For $P=\{x\in\mathbb{R}^n : Ax=b,\ x\geq 0\}$ with $A\in\mathbb{R}^{m\times n}$, $\operatorname{rank}(A)=m$

and $v\in P$, v is a vertex of P

$\Leftrightarrow$ v is a basic feasible solution

$\Leftrightarrow$ col of A_K linearly independent where $K=\{j : v_j>0\}$

$\Leftrightarrow$ $\exists B\subseteq\{1,\dots,n\}$ s.t. $|B|=m$, $B\supseteq K$, A_B invertible

Ex: $A=\begin{pmatrix}1&0&0&1&1\\0&1&0&2&0\\0&0&1&0&3\end{pmatrix}$ $b=\begin{pmatrix}3\\2\\0\end{pmatrix}$ $v=(2,0,0,1,0)$ $n=5$, $m=3$

$K=\{1,4\}\rightarrow$ col of A_K lin indep $\Rightarrow$ v basic feas. sol

Corresponding feasible bases: 134, 145

Why not 124? $A_{124}=\begin{pmatrix}1&0&1\\0&1&2\\0&0&0\end{pmatrix}$ not invertible $\Rightarrow$ not a basis

Feasible basis: $B \subseteq \{1,\dots,n\}$, $|B|=m$, A_B invertible and $A_B^{-1}b \geq 0$

$\longrightarrow$ basic feasible solution: $(x_B, x_N) = (A_B^{-1}b, 0)$

Simplex tableau: $$x_B = p + Qx_N$$ $$z = z_0 + r^T x_N$$

} same sol as $Ax=b$, $z=c^Tx$

Simplex Method: walks between feasible bases to improve obj. function

To find feasible basis solve aux. LP

$$\max -x_{n+1} - x_{n+2} - \dots - x_{n+m} \quad \text{s.t.} \quad Ax + \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} = b$$
$$x \geq 0, \ \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} \geq 0$$

always feasible, bounded, with $\{n+1,\dots,n+m\}$ as a feasible basis

Opt val $< 0 \Rightarrow$ original LP infeasible

Opt val $= 0 \Rightarrow$ original LP feasible

Solution $v \in \mathbb{R}^{n+m}$ with $v_{n+1} = \dots = v_{n+m} = 0$

$\rightarrow (v_1,\dots,v_n)$ feasible for original LP

From feasible basis B for original LP with

$$T(B): \quad x_B = p + Qx_N$$
$$z = z_0 + r^T x_N$$

$r \leq 0 \rightarrow$ done: $z_0 =$ opt val, $(x_B, x_N) = (p, 0)$ opt. solution

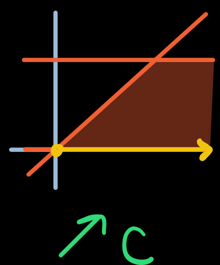
$r_i > 0 \rightarrow$ $q_i = i^{th}$ column of Q

If $q_i \geq 0$, LP unbounded, done

O.w. find j minimizing $\frac{-p_k}{(q_i)_k}$ s.t. $(q_i)_k < 0$.

Repeat w/ $B \cup \{i\} \setminus \{j\}$.

Ex: max $x_1 + x_2$ s.t. $x_1 - x_2 - x_3 = 0$, $x_2 + x_4 = 1$ $\quad x \geq 0$



$B = \{3, 4\}$

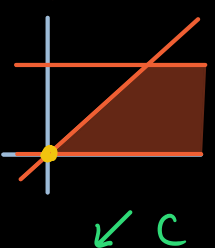
$$x_3 = x_1 - x_2$$
$$x_4 = 1 - x_2$$
$$z = \underline{\underline{x_1}} + x_2$$

all coeff of x_1 are ≥ 0
$\Rightarrow x_1$ unbounded!

Point $(x_1, 0, x_1, 1)$ feasible $\forall x_1 \geq 0$,

val of obj function $= x_1 \rightarrow \infty$ as $x_1 \rightarrow \infty$ $\Rightarrow$ unbounded

Ex: max $-x_1 - x_2$ s.t. $x_1 - x_2 - x_3 = 0$, $x_2 + x_4 = 1$ $\quad x \geq 0$



$B = \{3, 4\}$

$$x_3 = x_1 - x_2$$
$$x_4 = 1 - x_2$$
$$z = \underline{-x_1 - x_2}$$

all coeff ≤ 0
$\Rightarrow$ optimal

optimal val $= 0$

optimal solution: $(0, 0, 0, 1)$