Math 407 – Linear Optimization

Today: Other algorithms

Other algorithms for solving linear programs

Here we'll just sketch the main ideas.
See Ch 7 for more details

Input: $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$

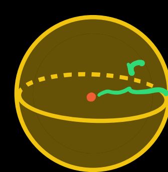
Input size $\approx n \cdot m \cdot L$ where $L = \log(\max\{|a_{ij}|, |b_i|, |c_j|\})$

1) The Ellipsoid Method

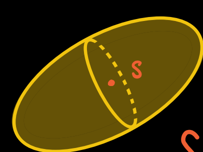
developped by Shor, Judin, Nemirovski (1970) for nonlinear opt

applied to give poly-time alg. for LP's by Khachyian (1979)

Ball of radius 1: $B_1 = \{x \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 \leq 1\}$



Ellipsoid: $\{Tx + s : x \in B_r\}$



s = center of ellipsoid

$= \{x \in \mathbb{R}^n : \|T^{-1}(x-s)\|_2^2 \leq 1\}$

$T: \mathbb{R}^n \to \mathbb{R}^n$ invertible linear map, $s \in \mathbb{R}^n$

Input: Polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$

Goal: Find a point $x \in P$ or decide P is empty.

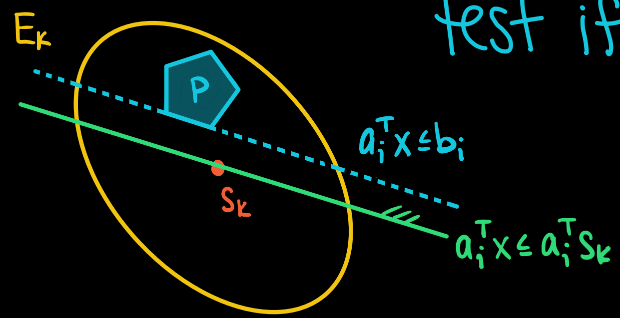
(Use to optimize: Find a pt in $P \cap \{x \in \mathbb{R}^n : c^T x \leq \lambda\}$)
for various λ

Simplifying assumptions:

- Know some ellipsoid E_0 containing P
- New goal: Find a point $x \in P$ or conclude that the volume of P is less than some predetermined $\varepsilon > 0$.

(Some preprocessing steps make these assumptions reasonable)

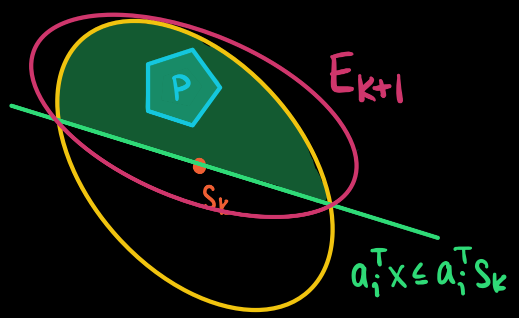
Main step: Starting from ellipsoid E_k containing P,



test if center s_k of E_k satisfies $As_k \le b$

If yes, then $s_k \in P$. STOP

If not, $a_i^T s_k > b_i$ for some defining ineq. $a_i^T x \le b_i$ of P.



Find minimum volume ellipsoid E_{k+1} containing $E_k \cap \{x \in \mathbb{R}^n : a_i^T x \le a_i^T s_k\}$

If volume of $E_{k+1} < \varepsilon$, STOP.

Otherwise, repeat with E_{k+1}

Crucial fact:

$$\frac{\mathrm{vol}(E_{k+1})}{\mathrm{vol}(E_k)} \le e^{-1/(2n+2)} \Rightarrow \mathrm{vol}(E_k) \le \mathrm{vol}(E_0) \cdot e^{\left(\frac{-k}{2(n+1)}\right)}$$

$\Rightarrow$ needs at most $2(n+1)\log\left(\frac{\mathrm{vol}(E_0)}{\varepsilon}\right)$ steps.

See §7.1 for more details

Warning: This can be slow in practice!

2) Interior Point Methods

Developped by Karmakar (1984)

Also polynomial time, competitive with simplex method (better on some classes of problems, worse on others)

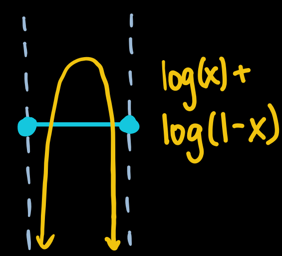
Take $P = \{x \in \mathbb{R}^n : Ax \leq b\}$.

Simplifying assumption: $\text{int}(P) = \{x \in \mathbb{R}^n : Ax < b\}$ nonempty

strict

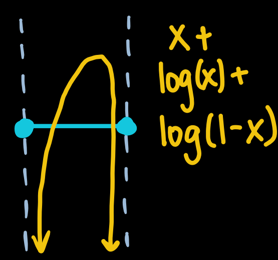
Logarithmic Barrier Function:

$$F(x) = \sum_{i=1}^{m} \log(b_i - a_i^T x)$$

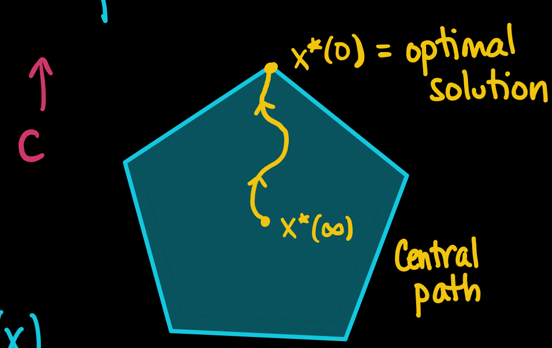


- defined and concave on $\text{int}(P)$
- goes to $-\infty$ as x approaches boundary of P

For $\lambda \geq 0$, $c^T x + \lambda F(x)$ also does this!



For $\lambda \geq 0$, let $x^*(\lambda)$ be the unique point in P maximizing $c^T x + F(x)$.



Central path: $\{x^*(\lambda) : \lambda > 0\}$

Idea: From $x^*(\infty)$ = pt maximizing $F(x)$ compute piecewise linear approximation to central path

(Newton approximations of $x_1^*(\lambda), \ldots, x_n^*(\lambda)$ as functions of λ)

Defined by the algebraic condition

$$\nabla\left(c^T x + \lambda F(x)\right) = 0$$

More details in §7.2