Math 407 – Linear Optimization

Today: Examples

Friday: Other algorithms

Monday: review

Midterm exam in class Wed. Nov. 1

(covering material through Oct 23)

Last time...

multiply eq. by -1 if needed to make $b_i \geq 0$

Original LP: $\max\ c^T x$ s.t. $x \geq 0,\ a_1^T x = b_1, \ldots, a_m^T x = b_m$

Aux LP: $\max\ -x_{n+1} - \ldots - x_{n+m}$ s.t. $(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \geq 0$

$a_1^T x + x_{n+1} = b_1, \ldots, a_m^T x + x_{n+m} = b_m$

$x_{n+1}, \ldots, x_{n+m}$ = auxiliary var. not in orig LP

Thm: Orig LP feasible $\Longleftrightarrow$ opt. val. of Aux LP is 0.

Ex: (original LP)

$\max\ x_1$ s.t.

$(x_1, x_2, x_3, x_4, x_5) \geq 0$

$x_1 + x_2 + x_3 = 2$

$-x_1 + x_2 + x_4 = -1$

$x_1 - x_2 + x_5 = -1$

$\rightarrow$ (Aux LP)

$\max\ -x_6 - x_7 - x_8$ s.t.

$(x_1, \ldots, x_8) \geq 0$

$x_1 + x_2 + x_3 + x_6 = 2$

$x_1 - x_2 - x_4 + x_7 = 1$

$-x_1 + x_2 - x_5 + x_8 = 1$

$B = \{6,7,8\}$ is feasible (corres. to $\bar{x} = (0,0,0,0,0,2,1,1)$)

$T(\{6,7,8\})$:

$$\begin{aligned}
x_6 &= 2 - x_1 - x_2 - x_3 \\
x_7 &= 1 - x_1 + x_2 + x_4 \\
x_8 &= 1 + x_1 - x_2 + x_5 \\
\hline
z &= -4 + \underline{x_1} + x_2 + x_3 - x_4 - x_5
\end{aligned}$$

x_1 enters

$\rightarrow x_7$ leaves

$T(\{1,6,8\})$

$$\begin{aligned}
x_1 &= 1 + x_2 + x_4 - x_7 \\
x_6 &= 1 - 2x_2 - x_3 - x_4 + x_7 \\
x_8 &= 2 + x_4 + x_5 - x_7 \\
\hline
z &= -3 + 2x_2 + \underline{x_3} - x_5 - x_7
\end{aligned}$$

x_3 enters

$\rightarrow x_6$ leaves

$T(\{1,3,8\})$

$$\begin{aligned}
x_1 &= 1 + x_2 + x_4 - x_7 \\
x_3 &= 1 - 2x_2 - x_4 - x_6 + x_7 \\
x_8 &= 2 + x_4 + x_5 - x_7 \\
\hline
z &= -2 - x_4 - x_5 - x_6
\end{aligned}$$

optimal val $= -2$

$\Rightarrow$ original LP infeasible

When orig. LP is infeasible, we can read off a certificate of infeasibility from last row of the simplex tableau of optimal solution (of Aux LP)

In Ex: last row means that

$$\left.\begin{aligned} x_1+x_2+x_3+x_6 &= 2 \\ x_1-x_2-x_4+x_7 &= 1 \\ -x_1+x_2-x_5+x_8 &= 1 \\ z &= -x_6-x_7-x_8 \end{aligned}\right\} \Rightarrow -x_6-x_7-x_8 = -2-x_4-x_5-x_6$$

$$\Rightarrow x_4+x_5 = -2+x_7+x_8$$

$$\Rightarrow x_4+x_5 = -2+(1-x_1+x_2+x_4)+(1+x_1-x_2+x_5) \quad (*)$$

If $(x_1,\dots,x_5)$ were feasible for original LP, then $x_4, x_5 \geq 0 \Rightarrow x_4+x_5 \geq 0$ and $(1-x_1+x_2+x_4)=0$, $(1+x_1-x_2+x_5)=0$

Plugging into $(*)$ gives $0 \leq x_4+x_5 = -2+0+0 = -2$ $\Rightarrow\Leftarrow$

Alternatively adding the last two equations of the original LP gives $x_4+x_5=-2$. With $x_4 \geq 0$, $x_5 \geq 0$, this is infeasible. The Auxiliary LP discovered this for us!

In general, if opt value of Aux. LP is $z_0 < 0$ and the last line in the simplex tableau is

$$z = z_0 + r_1 x_1 + \dots + r_{n+m} x_{n+m} \quad \text{with all } r_j \leq 0$$

Then for any $(x_1,\dots,x_{n+m})$ feasible for Aux LP,

$$-x_{n+1}-\dots-x_{n+m} = z_0 + r_1 x_1 + \dots + r_{n+m} x_{n+m}$$

$$\Rightarrow -r_1 x_1 - \dots - r_n x_n = z_0 + (r_{n+1}+1)\,x_{n+1} + \dots + (r_{n+m}+1)\,x_{n+m}$$

$$\Rightarrow -r_1 x_1 - \dots - r_n x_n = z_0 + (r_{n+1}+1)(b_1 - a_1^T x) + \dots + (r_{n+m}+1)(b_m - a_m^T x)$$

If $(x_1,\dots,x_n)$ is feasible for the original LP, then $x_1 \geq 0, \dots, x_n \geq 0$ and $a_1^T x = b_1, \dots, a_m^T x = b_m$

Plugging into eq above gives

$$\underbrace{-r_1 x_1 - \dots - r_n x_n}_{\substack{\geq 0 \text{ since} \\ -r_j \geq 0,\ x_j \geq 0}} = z_0 + (r_{n+1}+1)\underbrace{(b_1 - a_1^T x)}_{=0} + \dots + (r_{n+m}+1)\underbrace{(b_m - a_m^T x)}_{=0}$$

We've shown that if x is feasible for the original LP, then $z_0 \geq 0$. When $z_0 < 0$ this is a contradiction!

We'll explore more of these "certificates" when we talk about LP duality.