Math 407 – Linear Optimization

Today: Finding a feasible basis to start with

In many examples, we've been able to guess a (basic) feasible solution to start from.

Ex: max x_1+2x_3 s.t. $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

$$2x_1 - x_2 + 3x_3 \leq 2, \quad -x_1 + x_2 + 4x_3 \leq 6$$

Guess: Since RHS of ineq ≥ 0, we can guess $(x_1, x_2, x_3) = (0,0,0)$

What if there's not an obvious feasible basis?

Original LP: max $c^T x$ s.t. $Ax = b$, $x \geq 0$

with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

Multiplying equations by -1 if needed, we can take $b \geq 0$.

Idea: $Ax \leq b$ now feasible since $A\vec{0} = \vec{0} \leq b$

From this starting point, we can use the simplex method to look for a feasible solution with $Ax = b$.

Auxiliary LP:

$$\max \; -x_{n+1} - \ldots - x_{n+m} \quad \text{s.t. } \bar{A}\bar{x} = b, \; \bar{x} \geq 0$$

with $\bar{x} = (x_1, \ldots x_n, \underbrace{x_{n+1}, \ldots, x_{n+m}}_{\text{new}})^T$, $\bar{A} = (A \;\; I_m) \in \mathbb{R}^{m \times (n+m)}$

Note: $\bar{A}\bar{x} = Ax + \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix}$

Claim 1: $B = \{n+1, \dots, n+m\}$ is a feasible basis of Aux. LP.

Why? $\bar{A}\bar{x} = b$ has a unique sol with $x_1 = \dots = x_n = 0$

namely $x_1 = 0, \dots, x_n = 0,\ x_{n+1} = b_1, \dots, x_{n+m} = b_m$

Since $b \geq 0$, this is feasible

Claim 2: The value of the Aux. LP is $= 0$ if and only if original LP is feasible.

(Proof) ($\Leftarrow$) If $Av = b$, $v \geq 0$, then $\bar{x} = (v, 0)$ is feasible for Aux. LP with value 0.

($\Rightarrow$) Since $x_{n+1} \geq 0, \dots, x_{n+m} \geq 0$, $-x_{n+1} - \dots - x_{n+m} = 0$ implies that $x_{n+1} = \dots = x_{n+m} = 0$. So a point $v \in \mathbb{R}^{n+m}$ acheving optimal value has $v_{n+1} = \dots = v_{n+m} = 0$.

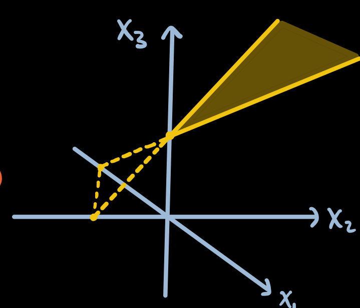
$\Longrightarrow v_{\{1,\dots,n\}}$ feasible for orignal LP.

Ex: LP: $\max c^T x$ s.t. $x_1 + x_2 = -1 \quad x_1 \geq 0, x_2 \geq 0$

$-x_1 - x_2 = 1$

Aux LP: $\max -x_3$ s.t. $-x_1 - x_2 + x_3 = 1$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$



$B = \{3\}$ feasible

$x_3 = 1 + x_1 + x_2$

$z = -1 - x_1 - x_2$

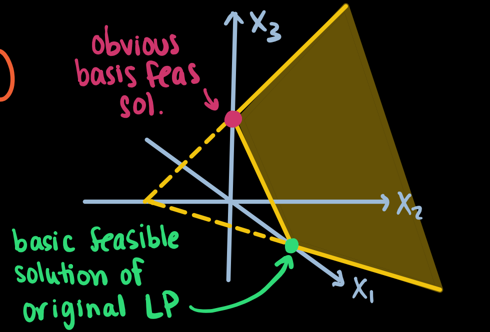
optimal value $= -1$

$\Rightarrow$ original LP infeasible

Ex: $\max c^T x$ s.t. $x_1 - x_2 = 1$, $x_1 \geq 0$, $x_2 \geq 0$

Aux LP: $\max -x_3$ s.t. $x_1 - x_2 + x_3 = 1$

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$



$B = \{3\}$ feasible (corres. to $(0,0,1)$)

$T(\{3\})$: $x_3 = 1 - x_1 + x_2$
$z = -1 + \underline{x_1} - x_2$

$\xrightarrow{\text{pivot}}$ x_1 enters, x_3 leaves

$T(\{1\})$ $x_1 = 1 + x_2 - x_3$
$z = 0 \quad - x_3$

opt val $= 0$
opt sol: $(1,0,0)$
gives feas. sol. $(x_1, x_2) = (1,0)$ of orig LP

Remark: If $v \in \mathbb{R}^{n+m}$ is a basic feas. sol. of Aux LP with $v_{n+1} = \dots = v_{n+m} = 0$, then v is indexed by a basis $B \subseteq \{1, \dots, n\}$.

$\Rightarrow v_{\{1,\dots,n\}} \in \mathbb{R}^n$ is a basic feas. sol. of orig. LP

(indexed by the same basis! $\bar{A}_B = A_B$ invertible, $\bar{A}_B^{-1} b = A_B^{-1} b \geq 0$)

The Full Simplex Method

Input: $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, choice of pivot rule

Output: "LP infeasible", "LP unbounded" or

$\max c^T x$ s.t. $Ax = b$, $x \geq 0$.

PHASE I (when there is no obvious feasible basis)

Scale rows of $(A \mid b)$ by ± 1 so that $b \geq 0$

Set up Aux. LP: $\max -x_{n+1} - \dots - x_{n+m}$ s.t. $Ax + \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} = b$, $\bar{x} \geq 0$

Starting from $B = \{n+1, \dots, n+m\}$, pivot until at optimal solution

- optimal value < 0 $\Rightarrow$ original LP infeasible
- optimal value = 0 $\Rightarrow$ original LP feasible, $v_{\{1,..,n\}}$ basic feas. sol. for orig. LP. with optimal solution $v \in \mathbb{R}^{n+m}$

<u>PHASE II</u>: Starting from feasible basis, pivot until stopped by optimality or unboundedness