Math 407 – Mathematica Demo

October 20, 2023

Example 1. Let's try the simplex algorithm on a larger example. Namely

$$\max\ x_1 + x_2 \quad \text{s.t.} \quad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_1 - x_2 \leq 1,\ x_2 - x_3 \leq 1,\ -x_1 + x_3 \leq 1$$

We add slack variables to get into equational form, giving

$$\begin{aligned} \max\ x_1 + x_2 \quad \text{s.t.} \quad (x_1, x_2, x_3, x_4, x_5, x_6) &\geq 0 \\ x_1 - x_2 + x_4 &= 1 \\ x_2 - x_3 + x_5 &= 1 \\ -x_1 + x_3 + x_6 &= 1 \end{aligned}$$

In the original LP, the point $(x_1, x_2, x_3) = (0, 0, 0)$ is feasible, so we see that $(0, 0, 0, 1, 1, 1)$ is feasible for the LP in equational form. The corresponding feasible basis is $\{4, 5, 6\}$, with tableau

$$\begin{array}{rl} \mathcal{T}(\{4,5,6\}): & \begin{array}{rclllll} x_4 &=& 1 &-\ x_1 &+\ x_2 & \\ x_5 &=& 1 & &-\ x_2 &+\ x_3 \\ x_6 &=& 1 &+\ x_1 & &-\ x_3 \\ \hline z &=& & \ \ x_1 &+\ x_2 & \end{array} \\ v = \ (0,0,0,1,1,1) & \end{array}$$

We can choose x_1 or x_2 to enter. Let's choose x_1. We increase x_1 while keeping $x_2 = x_3 = 0$. The maximum feasible value is $x_1 = 1$, which makes $x_4 = 0$. So x_4 leaves.

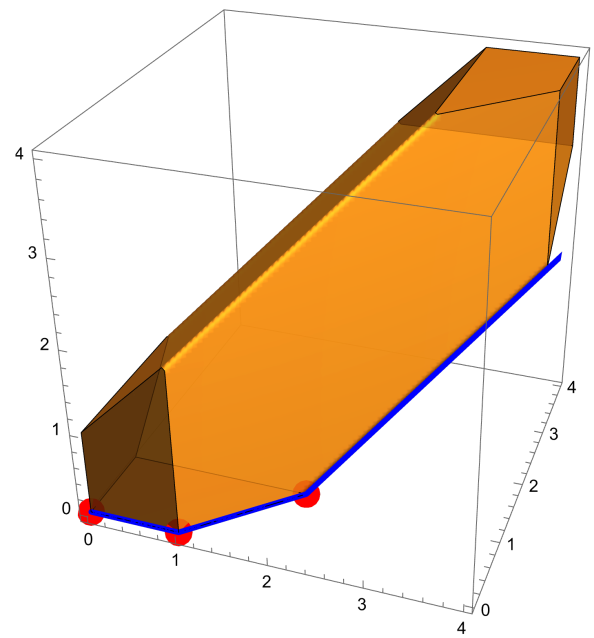
$$\begin{array}{rl} \mathcal{T}(\{1,5,6\}): & \begin{array}{rclllll} x_1 &=& 1 &+\ x_2 & &-\ x_4 \\ x_5 &=& 1 &-\ x_2 &+\ x_3 & \\ x_6 &=& 2 &+\ x_2 &-\ x_3 &-\ x_4 \\ \hline z &=& 2 &+\ 2x_2 & &-\ x_4 \end{array} \\ v = \ (1,0,0,0,1,2) & \end{array}$$

Only x_2 can enter. We increase x_2 while keeping $x_3 = x_4 = 0$. The maximum feasible value is $x_2 = 1$, which makes $x_5 = 0$. So x_5 leaves.

$$\begin{array}{rl} \mathcal{T}(\{1,2,6\}): & \begin{array}{rclllll} x_1 &=& 2 &+\ x_3 &-\ x_4 & \\ x_2 &=& 1 &+\ x_3 & &-\ x_5 \\ x_6 &=& 3 & &-\ x_4 &-\ x_5 \\ \hline z &=& 3 &+\ 2x_3 &-\ x_4 &-\ 2x_5 \end{array} \\ v = \ (2,1,0,0,0,3) & \end{array}$$

Only x_3 can enter. We increase x_3 while keeping $x_4 = x_5 = 0$. This is feasible for any $x_3 \geq 0$. We conclude that the LP is unbounded. Indeed, the point $(2 + x_3, 1 + x_3, x_3, 0, 0, 3)$ is feasible for all $x_3 \geq 0$ and the value of the objective funciton at this point is $3 + 2x_3$ which $\to \infty$ as $x_3 \to \infty$.

We can see the path that we took through our pivots on the plot of the feasible region using x_1, x_2, x_3 as parameters for our feasible region:



Example 2. [Chvátal, Problem 10.2] A furniture-manufacturing company makes bookcases, desks, chairs, and bedframes.

- A bookcase requires three hours of work, one unit of metal, and four units of wood, and brings in a net profit of $19.
- A desk requires two hours of work, one unit of metal, and three units of wood, and brings in a net profit of $13.
- A chair requires one hour of work, one unit of metal, and three units of wood, and brings in a net profit of $12.
- A bedframe requires two hours of work, one unit of metal, and four units of wood, and brings in a net profit of $17.
- Only 225 hours of labor, 117 units of metal, and 420 units of wood are available per day.

How should they use their resources to maximize the profit?
We model this problem as the LP

$$\max 19x_1 + 13x_2 + 12x_3 + 17x_4 \quad \text{s.t.} \quad \begin{aligned} (x_1, x_2, x_3, x_4, x_5, x_6, x_7) &\geq 0 \\ 3x_1 + 2x_2 + x_3 + 2x_4 + x_5 &= 225 \\ x_1 + x_2 + x_3 + x_4 + x_6 &= 117 \\ 4x_1 + 3x_2 + 3x_3 + 4x_4 + x_7 &= 420 \end{aligned}$$

where

$$\begin{aligned} x_1 &= \#\text{bookcases} & x_5 &= \#\text{unused hours} \\ x_2 &= \#\text{desks} & x_6 &= \#\text{unused units of metal} \\ x_3 &= \#\text{chairs} & x_7 &= \#\text{unused units of wood} \\ x_4 &= \#\text{bedframes} \end{aligned}$$

There is a natural feasible basis to start from – namely make nothing! The corresponding basis is $\{5, 6, 7\}$ (since these index the nonzero coordinates resulting from making no products).

$$\mathcal{T}(\{5,6,7\}): \quad \begin{array}{rclcrcrcrcr} x_5 &=& 225 &-& 3x_1 &-& 2x_2 &-& x_3 &-& 2x_4 \\ x_6 &=& 117 &-& x_1 &-& x_2 &-& x_3 &-& x_4 \\ x_7 &=& 420 &-& 4x_1 &-& 3x_2 &-& 3x_3 &-& 4x_4 \\ \hline z &=& & & 19x_1 &+& 17x_2 &+& 12x_3 &+& 17x_4 \end{array}$$

$$v = (0,0,0,0,225,117,420)$$

We see from the last line that at the current basic feasible solution, we are making no profit and that we can increase any nonbasic variable to increase the profit. Perhaps we try to start making bookcases, since those are the most profitable. That is, we choose x_1 to enter. We increase x_1 while keeping $x_2 = x_3 = x_4 = 0$. The maximum feasible value is $x_1 = 225/3 = 75$, which makes $x_5 = 0$. So x_5 leaves.

$$\mathcal{T}(\{1,6,7\}): \quad \begin{array}{rclcrcrcrcr} x_1 &=& 75 &-& 2x_2/3 &-& x_3/3 &-& 2x_4/3 &-& x_5/3 \\ x_6 &=& 42 &-& x_2/3 &-& 2x_3/3 &-& x_4/3 &+& x_5/3 \\ x_7 &=& 120 &-& x_2/3 &-& 5x_3/3 &-& 4x_4/3 &+& 4x_5/3 \\ \hline z &=& 1425 &+& x_2/3 &+& 17x_3/3 &+& 13x_4/3 &-& 19x_5/3 \end{array}$$

$$v = (75,0,0,0,0,42,120)$$

At this basic feasible solution, we are making 75 bookcases and a profit of \$1425. From the last line, we see that it could be profitable to start making one of the other products (while still keeping the number of unused hours equal to zero). Suppose we choose to start making chairs. That is, we choose x_2 to enter. We increase x_2 while keeping $x_3 = x_4 = x_5 = 0$. The maximum feasible value is $x_2 = 225/2$, which makes $x_1 = 0$. So x_1 leaves.

$$\mathcal{T}(\{2,6,7\}): \quad \begin{array}{rclcrcrcrcr} x_2 &=& 225/2 &-& 3x_1/2 &-& x_3/2 &-& x_4 &-& x_5/2 \\ x_6 &=& 9/2 &+& x_1/2 &-& x_3/2 & & &+& x_5/2 \\ x_7 &=& 165/2 &+& x_1/2 &-& 3x_3/2 &-& x_4 &+& 3x_5/2 \\ \hline z &=& 2925/2 &-& x_1/2 &+& 11x_3/2 &+& 4x_4 &-& 13x_5/2 \end{array}$$

$$v = (0, \tfrac{225}{2}, 0, 0, 0, \tfrac{9}{2}, \tfrac{165}{2})$$

We are now making only desks! The profit has increased from \$1425 to \$2925/2 = \$1462.50. We not yet at the optimal solution. We can choose x_3 or x_4 to enter. Let's choose x_3. We increase x_3 while keeping $x_1 = x_4 = x_5 = 0$. The maximum feasible value is $x_3 = 9$, which makes $x_6 = 0$. So x_6 leaves.

$$\mathcal{T}(\{2,3,7\}): \quad \begin{array}{rclcrcrcrcr} x_2 &=& 108 &-& 2x_1 &-& x_4 &-& x_5 &+& x_6 \\ x_3 &=& 9 &+& x_1 & & &+& x_5 &-& 2x_6 \\ x_7 &=& 69 &-& x_1 &-& x_4 & & &+& 3x_6 \\ \hline z &=& 1512 &+& 5x_1 &+& 4x_4 &-& x_5 &-& 11x_6 \end{array}$$

$$v = (0,108,9,0,0,0,69)$$

At the corresponding basic feasible solution, we're making 108 desks, 9 chairs, and a profit of \$1512. We not yet at the optimal solution. We can choose x_1 or x_4 to enter. Let's choose x_1. We increase x_1 while keeping $x_4 = x_5 = x_6 = 0$. The maximum feasible value is $x_1 = 108/2 = 54$, which makes $x_2 = 0$. So x_2 leaves.

$$\mathcal{T}(\{1,3,7\}): \quad \begin{array}{rcrcrcrcrcr} x_1 & = & 54 & - & x_2/2 & - & x_4/2 & - & x_5/2 & + & x_6/2 \\ x_3 & = & 63 & - & x_2/2 & - & x_4/2 & + & x_5/2 & - & 3x_6/2 \\ x_7 & = & 15 & + & x_2/2 & - & x_4/2 & + & x_5/2 & + & 5x_6/2 \\ \hline z & = & 1782 & - & 5x_2/2 & + & 3x_4/2 & - & 7x_5/2 & - & 17x_6/2 \end{array}$$

$$v = (54, 0, 63, 0, 0, 0, 15)$$

At the corresponding basic feasible solution, we're making 54 bookcases, 63 chairs, and a profit of \$1782. We not yet at the optimal solution. Only x_4 can enter. We increase x_4 while keeping $x_2 = x_5 = x_6 = 0$. The maximum feasible value is $x_4 = 30$, which makes $x_7 = 0$. So x_7 leaves.

$$\mathcal{T}(\{1,3,4\}): \quad \begin{array}{rcrcrcrcrcr} x_1 & = & 39 & - & x_2 & - & x_5 & - & 2x_6 & + & x_7 \\ x_3 & = & 48 & - & x_2 & & & - & 4x_6 & + & x_7 \\ x_4 & = & 30 & + & x_2 & + & x_5 & + & 5x_6 & - & 2x_7 \\ \hline z & = & 1827 & - & x_2 & - & 2x_5 & - & x_6 & - & 3x_7 \end{array}$$

$$v = (39, 0, 48, 30, 0, 0, 0)$$

At the corresponding basic feasible solution, we're making 39 bookcases, 48 chairs, 30 bed frames and a profit of \$1827. The coefficient of every variable in the last line is negative, so this is the (unique) optimal solution!