Today: The Simplex Method
(starting from a feasible basis)

The Simplex Method

LP: $\max c^T x$ s.t. $Ax = b$, $x \geq 0$

Pivot step: B with basic feas. sol v

$\tilde{B} = B \cup \{i\} \setminus \{j\}$ with basic feas. sol. $\tilde{v}$

with $c^T \tilde{v} \geq c^T v$ and $c^T \tilde{v} > c^T v$ if $\tilde{v} \neq v$.

Simplex method (starting from a feasible basis B)

Repeat pivot step until stopped by unboundednes or optimality

Q: Does this algorithm terminate?

There are only a finite number of bases ($\leq \binom{n}{m}$), so the only danger is if we come back to a basis

Bases: $B^{(1)} \to B^{(2)} \to \dots \to B^{(k)} \to B^{(1)}$

basic feas. sol: $v^{(1)} \to v^{(2)} \to \dots \to v^{(k)} \to v^{(1)}$

obj. function can't go down

$\Rightarrow c^T v^{(1)} \leq c^T v^{(2)} \leq \dots \leq c^T v^{(k)} \leq c^T v^{(1)}$

$\Rightarrow c^T v^{(1)} = c^T v^{(2)} = \dots = c^T v^{(k)}$

$\Rightarrow v^{(1)} = v^{(2)} = \dots = v^{(k)}$, since obj. function didn't improve

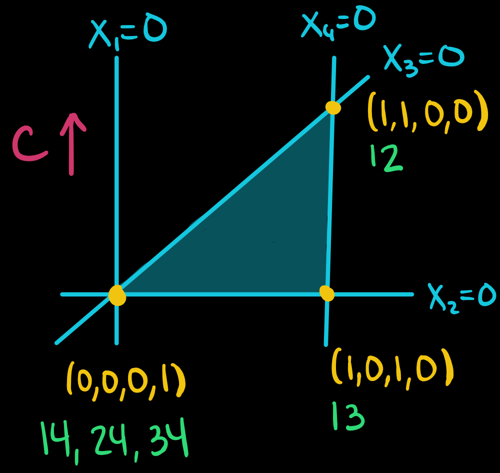
DANGER: A basic feasible solution can have more than n-m coord. equal to zero

$\Rightarrow$ corresponds to more than one basis.

We need to avoid cycling around different representations of the same point in $\mathbb{R}^n$.

Ex: max x_2 s.t.

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad x \geq 0$$



$B=\{3,4\}$ $\quad$ $x_3 = x_1 - x_2$

$T(B)$ $\quad$ $x_4 = 1 - x_1$

$z = \underline{x_2}$

Choose x_2 to enter (keep $x_1=0$)

$x_3 = -x_2 \geq 0$ → max feas. val of $x_2 = 0$

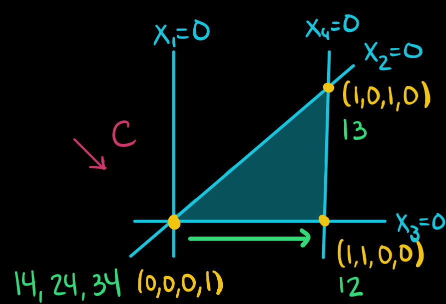
$x_4 = 1 \geq 0$ ⇒ makes $x_3=0$

(we can't increase x_2 while keeping $x_1=0$)

$B=\{2,4\}$ $\quad$ $x_2 = x_1 - x_3$

$x_4 = 1 - x_1$

$z = \underline{x_1} - x_3$



Choose x_1 to enter (keep $x_3=0$)

$x_2 = x_1 \geq 0$ ⇒ max feas val of $x_1 = 1$

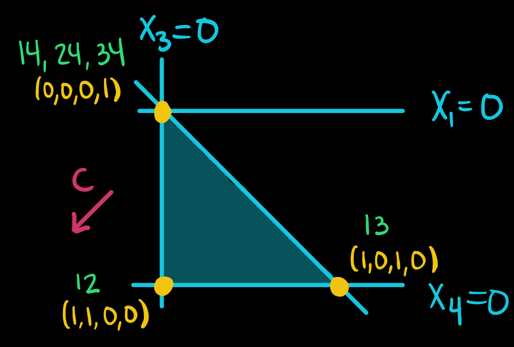
$x_4 = 1 - x_1 \geq 0$ ⇒ makes $x_4 = 0$

$B=\{1,2\}$ $\quad$ $x_1 = 1 - x_4$

$x_2 = 1 - x_3 - x_4$

$z = \underline{1 - x_3 - x_4}$

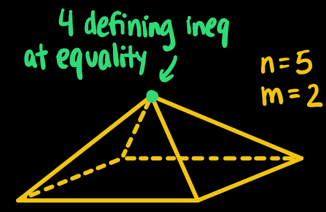
Optimal sol acheived by (1,1,0,0)



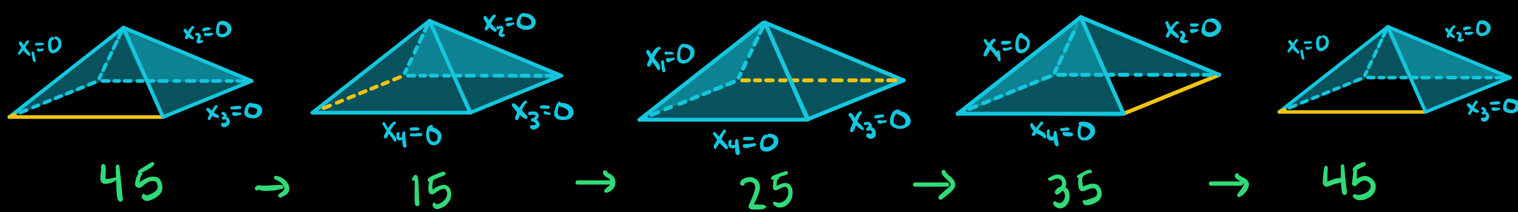
Here there's a redundant inequality, but

(1) discovering if an inequality $c^T x \leq \lambda$ is implied by others can be as hard as solving an LP

(2) Even w/ no redundant inequalities vertices can satisfy more than $n-m$ inequalities w/ equality.



This can happen with some natural pivot rules (⇒ algorithm gets stuck in a loop, never terminates)



Some pivot rules avoid this.

Bland's rule: Whenever there is a choice (of entering or leaving variable) choose that w/ the smallest index (also called lexicographic)

Thm: The simplex method w/ Bland's pivot rule does not cycle (never comes back to the same feasible basis twice).

(See Thm 5.8.1 for proof)

Cor: If the LP is feasible and bounded, the simplex method w/ Bland's rule terminates (necessarily at an optimal solution)

Does it terminate quickly?

In practice: usually terminates in ~3m pivot steps

In theory: not always! (Klee-Minty 1972)

at UW!

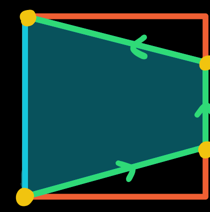
Fix $\varepsilon \in (0, \frac{1}{2})$ and consider LP(d):

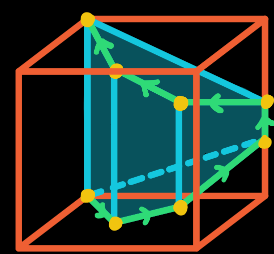
max x_d s.t.

$$\left.\begin{aligned} 0 \le\ & x_1 \le 1 \\ \varepsilon x_1 \le\ & x_2 \le 1 - \varepsilon x_1 \\ & \vdots \\ \varepsilon x_{d-1} \le\ & x_d \le 1 - \varepsilon x_{d-1} \end{aligned}\right\}$$

2d inequalities in d variables

→ n=3d, m=2d in equational form





Using "largest coefficient" pivot rule, simplex method takes $2^d - 1$ iterations to terminate.

Visits all vertices of a skewed n-dim'l cube

<u>Open Question</u>: Does there exist a pivot rule and polynomial p(n,m) so that the simplex simplex method solves any LP with m equations, n variables in p(n,m) steps?